QFT II

homework XIII (Dec. 26)

- **At the head of your report, please write your name, student ID number and a list of problems that you worked on in a report (like "II-1, II-3, IV-2").**

1. **More about path integral of a fermionic system** [B]

In an ordinary (bosonic) quantum mechanical system, its partition function for its canonical ensemble is expressed by parh integral

$$Z = \int \mathcal{D}p\mathcal{D}q \exp\left[-\int_0^\beta dt\,(H(p,q) - ip\partial_\tau q)\right] \tag{1}$$

with the periodic boundary condition

$$q(t = -i\beta) = q(t = 0). \tag{2}$$

In this homework problem, let us work out the analogue for a two-state (fermionic) quantum mechanical system. In the lecture, we have learned how to express time evolution of wavefunctions of this two state system in term of path integral; following the notation used in the lecture, where a state $|0\rangle c_1 + |1\rangle c_1$ corresponds to a wavefunction $\Psi(\bar\theta) = c_0 + \bar\theta c_1$, evolution in the negative imaginary direction is given by

$$\Psi_{\text{fin}}(\bar\theta_N) = \int d\bar\theta_{N-1}d\theta_{N-1}\cdots d\bar\theta_0 d\theta_0 e^{\left[\sum_{k=0}^N \bar\theta_k\theta_k - \sum_{k=0}^{N-1}\bar\theta_k\theta_k - (\Delta\tau)\sum_{k=1}^N H(\theta_{k-1},\bar\theta_k)\right]}\Psi_{\text{in}}(\bar\theta_0), \tag{3}$$

where $\Delta\tau = \beta/N$.

(a) Now, by using $Z = \mathrm{tr}[e^{-\beta H}] = \langle 0|e^{-\beta H}|0\rangle + \langle 1|e^{-\beta H}|1\rangle$, verify that

$$Z = \int d\bar\theta_N d\bar\theta_{N-1}\cdots d\bar\theta_0 d\theta_0 e^{[\cdots]}(\bar\theta_N + \bar\theta_0). \tag{4}$$

(b) For an arbitrary function $f(\bar\theta_N, \bar\theta_0)$ depending on Grassmann coordinates $\bar\theta_N$ and $\bar\theta_0$, verify that

$$\int d\bar\theta_N(\bar\theta_N + \bar\theta_0)f(\bar\theta_N,\bar\theta_0) = f(-\bar\theta_0,\bar\theta_0). \tag{5}$$

[This means that the factor $(\bar\theta_N + \bar\theta_0)$ inserted in a Grassmann integral can be regarded as something like a delta function $\delta(\bar\theta_N + \bar\theta_0)$.]

(c) [not a problem] By combining both, we see that the partition function of the canonical ensemble of the two state (fermionic) quantum mechanical system is

$$Z = \int \mathcal{D}\bar\theta\mathcal{D}\theta \exp\left[\bar\theta\partial_\tau\theta - \int d\tau H(\theta,\bar\theta)\right] \tag{6}$$

with the **anti**-periodic boundary condition

$$\bar\theta(t = -i\beta) = -\bar\theta(t = 0). \tag{7}$$

(d) Compute

$$\int d\theta d\bar{\theta} d\theta' d\bar{\theta}' \exp\left[(\bar{\theta}, \bar{\theta}') \begin{pmatrix} -m & p \\ p & -m \end{pmatrix} \begin{pmatrix} \theta \\ \theta' \end{pmatrix} \right]. \tag{8}$$

Contrary to the case with a boson, we see a positive power of $(p^2 - m^2)$.

(e) We have discussed the partition function $Z = \mathrm{tr}(e^{-\beta H})$ of a free boson. Write up a discussion in the case of a free Dirac fermion, using the two following equations:

$$\ln(Z) = 2 \sum_{n\in\mathbb{Z}} V_d \int \frac{d^d k}{(2\pi)^d} \ln\left(E_{\vec{k}}^2 + (2\pi T(n+1/2))^2 \right) \tag{9}$$

and

$$\ln(Z) = 4V_d \int \frac{d^d k}{(2\pi)^d} \left(\frac{\beta E_{\vec{k}}}{2} + \ln(1 + e^{-\beta E_{\vec{k}}}) \right). \tag{10}$$

[remark: the first term $+\beta E/2$ has the sign opposite from that in the case of a scalar field. So, in a combination of four real scalar fields and one Dirac fermion (also in a combination of one complex scalar and one Weyl fermion), the first terms cancel. This is due to supersymmetry. There is no such cancellation among the second terms (thermal contributions), however.]

2. **Chemical potential** [B]

One can switch from the canonical ensemble of a quantum system to its grand canonical ensemble by modifying the Hamiltonian H to $H - \sum_i \mu_i N_i$, where the label i runs over (a subset of) all the conserved numbers (charges) of the system of one's interest. N_i is the Noether charge of a U(1) symmetry, and μ_i the corresponding chemical potential. In the case of the Standard Model of particle physics, for example, the lepton number and the baryon number are conserved charges.[1] When an effective theory with much lower energy scale is considered, the number of atoms of various kinds (labeled by i) may be conserved separately.

(a) **relativisistic boson case** (cf. homework I-2 for non-relativistic cases (fermionic/bosonic))
Suppose that an effective theory with a relativistic complex boson is given by the Hamiltonian:

$$H = \int d^3x \pi^\dagger \pi + (\nabla\Phi)^*(\nabla\Phi) + V(|\Phi|^2), \tag{11}$$

where π is the canonical conjugate momentum of a complex scalar field Φ. First, write down the Noether charge $N = \int d^3x J^0$ in terms of π and Φ. Secondly, carry out the Gaussian integral with respect to π and π^* in

$$Z = \int \mathcal{D}\pi\mathcal{D}\pi^*\mathcal{D}\Phi\mathcal{D}\Phi^* \exp\left[-\int d\tau (H - \mu N) + i \int d\tau \left(\pi(\partial_\tau \Phi^*) + \pi^*(\partial_\tau \Phi) \right) \right], \tag{12}$$

to see how the Lagrangian is modified.

[1] when the tiny neutrino masses and non-perturbative electroweak effects are ignored, to be more precise.

3. **Thermal mass** [C]
   Compute the thermal mass in a system that you like, including the coefficient. [e.g., ϕ^4 theory, relativistic complex boson coupled to a U(1) gauge field, non-relativistic fermion coupled to a U(1) gauge field with a non-zero chemical potential, etc.]