Quantum Field Theory II

§1. Introduction: What is QFT (for)?

☆ Quantum theory of dynamical systems on (space) x (time)

- QED, QCD. $\mathcal{L} = -\frac{1}{4g^2} F^a_{\mu\nu} F^{a\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi$
- the Standard Model of particle physics
- electron – phonon system
- nucleon – pion system
- spin system
- $\mathcal{L} = \phi^\dagger \left(i\partial_t + \frac{\partial_x^2}{2m} - V(x) \right)\phi - f(|\phi|^2)$ etc.

☆ A theoretical framework for description/computation of processes where the # of particles is not preserved.

examples

$$\left(\begin{array}{l} e^+ + e^- \to 2\gamma, \quad e^+ + e^- \to q + \bar{q} \ \text{(two jets)} \\ g + g \to h + t + \bar{t}, \quad q + \bar{q} \to q + \bar{q} + h \\ p + p \ (\text{or } \bar{p}) \to \text{many hadrons} \ \ldots \end{array} \right.$$

☆ Quantum theory of many-body systems.

A system of N fermions with

$$H = \sum_{i=1}^{N} \left(-\frac{1}{2m} \partial_{x_i}^2 + \varphi(x_i) \right) + \sum_{i \neq j}^{N} \frac{c}{|x_i - x_j|}$$

is equivalent to a QFT with

$\psi(x) := \sum_n \psi_n(x)\, a_n$ annihilation operator

$\psi^\dagger(x) = \sum_n \psi_n^*(x)\, a_n^\dagger$ creation operator

$$\{ a_m, a_n^\dagger \} = \delta_{m,n} \qquad \{ \psi(x), \psi^\dagger(y) \} = \delta(x-y)$$

$$H = \int dx\, \psi^\dagger(x) \left(-\frac{\partial_x^2}{2m} + \varphi(x) \right) \psi(x) + \int dx \int dy\, \psi^\dagger(y) \psi^\dagger(x) \frac{c}{|x-y|} \psi(x) \psi(y).$$

— What is the ground state like?
— What is the excitation spectrum like? (when interactions are turned on)

— What are operator matrix elements like? (expectation values)

A side remark:

- The wave function in a quantum mechanical many-body system

$$\Psi(x_1, \cdots, x_N) = \frac{1}{\sqrt{N!}} \begin{vmatrix} \psi_{n_1}(x_1) & \cdots & \psi_{n_1}(x_N) \\ \vdots & \ddots & \vdots \\ \psi_{n_N}(x_1) & \cdots & \psi_{n_N}(x_N) \end{vmatrix}$$

of a state $a_{n_1}^\dagger a_{n_2}^\dagger \cdots a_{n_N}^\dagger |0\rangle$

is an operator matrix element $\langle 0 | \psi(x_1) \cdots \psi(x_N) | n_1, n_2, \cdots, n_N \rangle$.

- We do not have the "probability distribution" interpretation except in a system with a particle number conservation but without pair annihilation/creation.

★ General theory constraining quantum systems on (space × time) with locality and causality

- The Hilbert space does not always look like a Fock space.

  (A particle picture is not always available as a good approximation.)

- A Lagrangian is not always available.

- What can we learn about operator matrix elements, operator correlation functis, or the list of operators to begin with in such a system?

§2. S-matrix etc. and how to compute them.

§2.1. S-matrix, decay rate and cross section

Think of a case where interactions are turned on after $t=T_-$ and switched off by $t=T_+$.

H: the full Hamiltonian

H_0: the bilinear part of H_0 $\qquad H = H_0 + V$

example (QED) $\qquad H_0 = -\bar{\psi}(i\gamma^i\partial_i - m)\psi + (\text{photon})$

$$V = \bar{\psi}(\gamma^i e A_i + \gamma^0 e A_0)\psi$$

The Hilbert space of the free theory is generated by Fock states

$$\left\{ |0\rangle,\ a^\dagger_{n,\vec{p}}|0\rangle,\ a^\dagger_{n_1,\vec{p}_1} a^\dagger_{n_2,\vec{p}_2}|0\rangle,\ \cdots \right\} \quad \text{at } t=t_*.$$

For the interacting theory ($H_0 \rightsquigarrow H \rightsquigarrow H_0$), we can use the in-states

$$e^{\left(-i\int_{T_-}^{t_*} dt' H\right)} e^{-iH_0(T_- - t_*)} \text{ on } \left\{ |0\rangle,\ a^\dagger_{n,\vec{p}}|0\rangle,\ a^\dagger_{n_1,\vec{p}_1} a^\dagger_{n_2,\vec{p}_2}|0\rangle,\ \cdots \right\}$$

as a basis of the Hilbert space.

The "in-states" are denoted by

$$|\Omega\rangle,\quad \underset{=|n,\vec{p}\rangle^{in}}{a^\dagger_{n,\vec{p}}|\Omega\rangle} \quad \underset{=|n_1,\vec{p}_1;n_2,\vec{p}_2\rangle^{in}}{a^\dagger_{n_1,\vec{p}_1} a^\dagger_{n_2,\vec{p}_2}|\Omega\rangle}$$

We can also take another basis:

the out-states

$$e^{\left(i\int_{T_+}^{t_*} dt' H\right)} e^{-iH_0(T_+ - t_*)} \text{ on } \left\{ |0\rangle,\ a^\dagger_{n,\vec{p}}|0\rangle,\ a^\dagger_{n_1,\vec{p}_1} a^\dagger_{n_2,\vec{p}_2}|0\rangle,\ \cdots \right\}.$$

The "in-states" and "out-states" are normalized so that

$$\boxed{\begin{aligned} {}^{in}\langle n_1, \vec{p}_1 | n_2, \vec{p}_2 \rangle^{in} &= \delta_{n_1, n_2} (2E_{\vec{p}_1}) \delta^3(\vec{p}_1 - \vec{p}_2) (2\pi)^3 \\ {}^{out}\langle n_1, \vec{p}_1 | n_2, \vec{p}_2 \rangle^{out} &= \delta_{n_1, n_2} (2E_{\vec{p}_1}) \delta^3(\vec{p}_1 - \vec{p}_2) (2\pi)^3 \end{aligned}}$$

in relativistic situations. (if non-rela. ${}^{in}\langle \vec{p} | \vec{q} \rangle^{in} = \delta^3(\vec{p} - \vec{q})(2\pi)^3$ is more conventional.)

The S-matrix is defined by

$$\boxed{S_{\beta\alpha} := {}^{out}\langle \beta | \alpha \rangle^{in},}$$

α, β: Fock states

which measures the difference between the in-state basis and the out-state basis.

The truely scattering part of the S-matrix is in. $S_{\beta\alpha} = \mathbb{1}_{\beta\alpha} + (2\pi)^4 \delta^4(p_{out} - p_{in})\, i\mathcal{M}_{\beta\alpha}$.

For a single particle : decay rate

$$\boxed{d\Gamma = \frac{1}{(2E_{in})} \prod_{i=1}^{N_f} \left[\frac{d^3\vec{p}_i}{(2\pi)^3} \frac{1}{2E_{\vec{p}_i}} \right] (2\pi)^4 \delta^4(p_f - p_{in}) |\mathcal{M}|^2}$$

Lorentz invariant

A boosted particle has a longer lifetime.

For a pair of colliding particles : cross section

$$\boxed{d\sigma = \frac{1}{(2E_1)(2E_2)(\vec{v}_1 - \vec{v}_2)} \prod_{i=1}^{N_f} \left[\frac{d^3\vec{p}_i}{(2\pi)^3} \frac{1}{2E_{\vec{p}_i}} \right] (2\pi)^4 \delta^4(p_f - p_{in}) |\mathcal{M}|^2}$$

$|\vec{v}_1 - \vec{v}_2| := \left| \frac{p_1^z}{E_1} - \frac{p_2^z}{E_2} \right|$ when the two particles are moving ~~in the dire~~ along the z-axis.

boost invariant.

Q: Verify that $[\Gamma] = +1$ and $[\sigma] = -2$ for arbitrary N_f by dimension counting.

Q: Verify that $E_1 E_2 |\vec{v}_1 - \vec{v}_2|$ is boost invariant (along the collision axis).

Notes

$\frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2E_{\vec{p}}}$ is boost invariant.

This is because $\begin{pmatrix} p_z' \\ E' \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} p_z \\ E \end{pmatrix}$, and $\frac{dp_z'}{E'} = \frac{dp_z\, \gamma\left(1 + \frac{p_z}{E}\beta\right)}{E'} = \frac{dp_z}{E}$.

$(2E_{\vec{p}})\,\delta^3(\vec{p} - \vec{q})$ is also invariant.

A side remark

π^+ : $m \simeq 140\,\mathrm{MeV}$, $\tau \simeq 2.6 \times 10^{-8}\,s$ $\quad \pi^+ \to \mu^+ + \nu_\mu$ (2-body)

μ^+ : $m \simeq 105\,\mathrm{MeV}$, $\tau \simeq 2.2 \times 10^{-6}\,s$ $\quad \mu^+ \to e^+ + \nu_e + \bar{\nu}_\mu$ (3-body)

§2.2. LSZ reduction formula

(LSZ : Lehmann – Symanzik – Zimmermann)

$\boxed{\S 2.2.1}$

Consider a time-ordered correlation function

$$\langle\Omega| T\{\mathcal{O}_1(x_1)\,\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\}|\Omega\rangle \quad \text{——} (*)$$

of Heisenberg picture operators.

It can be re-written as

$$(*) = \langle\Omega|\, e^{\left(i\int_{t_*}^{T_+} H\right)}\, e^{-iH_0\cdot(T_+-t_*)}\, T\left\{\mathcal{O}_{1,I}(x_1)\cdots\mathcal{O}_{n,I}(x_n)\exp\left[-i\int_{T_-}^{T_+} V_I\right]\right\}$$
$$e^{-iH_0(t_*-T_-)}\, e^{\left(-i\int_{t_*}^{T_-} H\right)}\,|\Omega\rangle . \quad \text{——} (**)$$

using the interaction-picture operators $\mathcal{O}_I(x)$, $V_I(x)$ and a property

$$e^{-iH_0(t_*-t_f)}\, e^{\left(-i\int_{t_i}^{t_f} H\right)}\, e^{-iH_0(t_i-t_*)} = T\exp\left[-i\int_{t_i}^{t_f} dt'\, V_I(t')\right] .$$

Noting that the normalization constant Z in

$$e^{\left(-i\int_{T_-}^{t_*} H\right)}\, e^{-iH_0(T_--t_*)}\,|0\rangle =: Z\,|\Omega\rangle$$
$$e^{-iH_0(t_*-T_-)}\, e^{\left(-i\int_{t_*}^{T_-} H\right)}\,|\Omega\rangle = Z^{-1}|0\rangle$$

satisfies

$$|Z|^2 = |Z|^2\langle\Omega|\Omega\rangle = \langle 0|\, e^{iH_0(T_+-t_*)}\, e^{\left(-i\int_{T_-}^{T_+} H\right)}\, e^{-iH_0(T_--t_*)}\,|0\rangle$$
$$= \langle 0| T\{\exp\left[-i\int_{T_-}^{T_+} V_I\right]\}|0\rangle ,$$

we arrive at a relation

$$\boxed{\langle\Omega| T\{\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\}|\Omega\rangle = \frac{\langle 0| T\{\mathcal{O}_{1,I}(x_1)\cdots\mathcal{O}_{n,I}(x_n)\exp\left[-i\int_{T_-}^{T_+} V_I\right]\}|0\rangle}{\langle 0| T\{\exp\left[-i\int_{T_-}^{T_+} V_I\right]\}|0\rangle} .}$$

One can use the Feynman rule to compute

$$\langle 0|T\{\mathcal{O}_{1,I}(x_1)\cdots\mathcal{O}_{n,I}(x_n)\exp[-i\int_{T_-}^{T_+} V_I]\}|0\rangle \quad \text{and}$$

$$\langle 0|T\{\exp[-i\int_{T_-}^{T_+} V_I]\}|0\rangle.$$

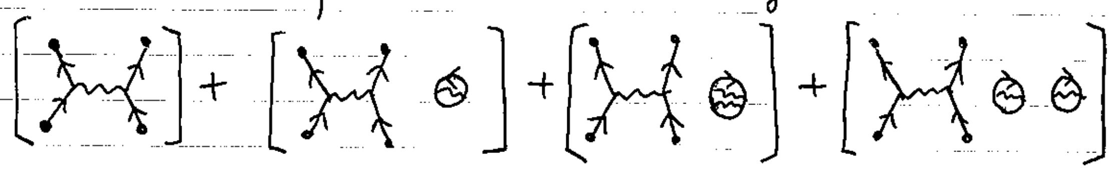
Example: for a relativistic real scalar field $\phi(x)$

$$\left(\langle 0|T\{\phi_I(x)\,\phi_I(y)\}|0\rangle = \int\frac{d^4p}{(2\pi)^4}\frac{i\,e^{-ip\cdot(x-y)}}{(p^2-m^2+i\varepsilon)}. \right.$$

$$\left. \text{use} \quad \phi_I(x)=\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec p}}}\left(a_{\vec p}\,e^{-ip\cdot x}+a_{\vec p}^\dagger\,e^{ip\cdot x}\right) \quad \text{and} \quad [a_{\vec p}, a_{\vec q}^\dagger]=(2\pi)^3\delta^3(\vec p-\vec q) \right)$$

Two important properties

- Vacuum bubbles factorize and cancel agains the denominator:



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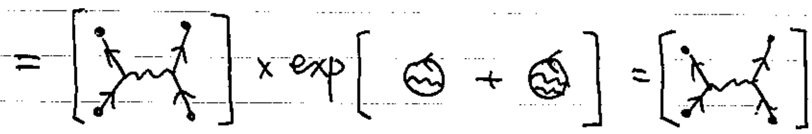
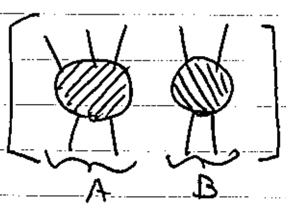


$\times\langle 0|T\{\exp[-i\int_{T_-}^{T_+}V_I]\}|0\rangle$

(see Peskin-Schroeder §4.4)

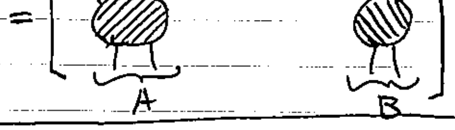
- Cluster decomposition

Feynman graphs (contributions to the correlation function) of the form



(not fully connected) are regarded as separate mutually non-interfering processes.

note also



$$\boxed{\text{Therefore,}\quad \langle 0|T\{\mathcal{O}_{1,I}(x_1)\cdots\mathcal{O}_{n,I}(x_n)\exp[-i\int_{T_-}^{T_+}V_I]\}|0\rangle_{\text{fully conn.}} \quad \text{is what we are after.}}$$