§2.2. LSZ reduction formula

(LSZ : Lehmann – Symanzik – Zimmermann)

§2.2.1

Consider a time-ordered correlation function

$$\langle\Omega|T\{\mathcal{O}_1(x_1)\,\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\}|\Omega\rangle \quad \text{———} (*)$$

of Heisenberg picture operators.

It can be re-written as

$$(*) = \langle\Omega|\, e^{\left(i\int_{t_*}^{T_+} H\right)}\, e^{-iH_0(T_+-t_*)}\, T\left\{\mathcal{O}_{1,I}(x_1)\cdots\mathcal{O}_{n,I}(x_n)\exp\left[-i\int_{T_-}^{T_+} V_I\right]\right\}$$
$$e^{-iH_0(t_*-T_-)}\, e^{\left(-i\int_{t_*}^{T_-} H\right)}|\Omega\rangle. \quad \text{———} (**)$$

using the interaction-picture operators $\mathcal{O}_I(x)$, $V_I(x)$ and a property

$$e^{-iH_0(t_*-t_f)}\, e^{\left(-i\int_{t_i}^{t_f} H\right)}\, e^{-iH_0(t_i-t_*)} = T\exp\left[-i\int_{t_i}^{t_f} dt'\, V_I(t')\right].$$

Noting that the normalization constant Z in

$$e^{\left(-i\int_{T_-}^{t_*} H\right)}\, e^{-iH_0(T_--t_*)}|0\rangle =: Z|\Omega\rangle$$
$$e^{-iH_0(t_*-T_-)}\, e^{\left(-i\int_{t_*}^{T_-} H\right)}|\Omega\rangle = Z^{-1}|0\rangle$$

satisfies

$$|Z|^2 = |Z|^2\langle\Omega|\Omega\rangle = \langle 0|e^{iH_0(T_+-t_*)}\, e^{\left(-i\int_{T_-}^{T_+} H\right)}\, e^{-iH_0(T_--t_*)}|0\rangle$$
$$= \langle 0|T\left\{\exp\left[-i\int_{T_-}^{T_+} V_I\right]\right\}|0\rangle,$$

we arrive at a relation

$$\boxed{\langle\Omega|T\{\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\}|\Omega\rangle = \frac{\langle 0|T\left\{\mathcal{O}_{1,I}(x_1)\cdots\mathcal{O}_{n,I}(x_n)\exp\left[-i\int_{T_-}^{T_+} V_I\right]\right\}|0\rangle}{\langle 0|T\left\{\exp\left[-i\int_{T_-}^{T_+} V_I\right]\right\}|0\rangle}.}$$

One can use the Feynman rule to compute

$$\langle 0|T\{\mathcal{O}_{1,I}(x_1)\cdots\mathcal{O}_{n,I}(x_n)\exp[-i\int_{T_-}^{T_+}V_I]\}|0\rangle \quad \text{and}$$

$$\langle 0|T\{\exp[-i\int_{T_-}^{T_+}V_I]\}|0\rangle .$$

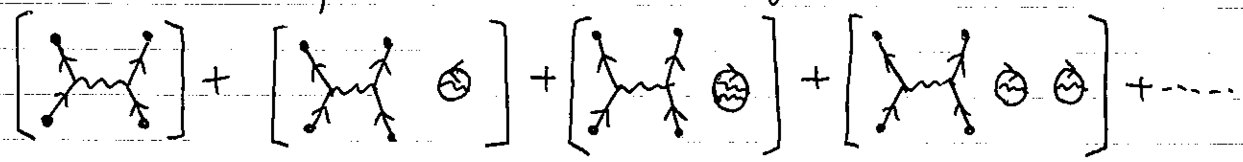
Example: for a relativistic real scalar field $\phi(x)$

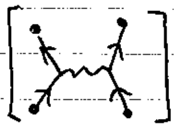
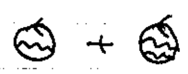
$$\left(\langle 0|T\{\phi_I(x)\,\phi_I(y)\}|0\rangle = \int\frac{d^4p}{(2\pi)^4}\frac{i\,e^{-ip\cdot(x-y)}}{(p^2-m^2+i\varepsilon)} . \right.$$

use $\phi_I(x) = \int\frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left(a_{\vec{p}}\,e^{-ip\cdot x}+a_{\vec{p}}^{\dagger}\,e^{ip\cdot x}\right)$ and $[a_{\vec{p}},a_{\vec{q}}^{\dagger}]=(2\pi)^3\delta^{3}(\vec{p}-\vec{q})$)

Two important properties

- Vacuum bubbles factorize and cancel agains the denominator:

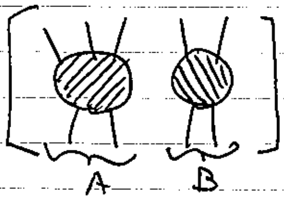


$=$  $\times \exp[$  $] =$  $\times\langle 0|T\{\exp[-i\int_{T_-}^{T_+}V_I]\}|0\rangle$

(see Peskin-Schroeder §4.4)

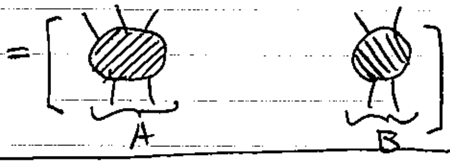
- Cluster decomposition

Feynman graphs (contributions to the correlation function) of the form



(not fully connected) are regarded as separate mutually non-interfering processes.

note also



$$\boxed{\text{Therefore, } \langle 0|T\{\mathcal{O}_{1,I}(x_1)\cdots\mathcal{O}_{n,I}(x_n)\exp[-i\int_{T_-}^{T_+}V_I]\}|0\rangle_{\text{fully conn.}} \text{ is what we are after.}}$$

§2.2.2 ~~Källen-Lehmann spectral representation & SD eq~~

How is the S-matrix related to time-ordered correlation functions?

Think of $\int d^4y \int d^4x \; e^{ip\cdot y} e^{-iq\cdot x} \langle \Omega | T\{A(y) B(x)\} | \Omega \rangle$. ——— $(\ast)$

Due to the translational symmetry, $P^\mu |\Omega\rangle = 0$, so

↑ Noether charge of translation.

$$\langle \Omega | T\{A(y) B(x)\} | \Omega \rangle$$

$$= \langle \Omega | e^{-iP\cdot a} T\{A(y) B(x)\} e^{iP\cdot a} | \Omega \rangle = \langle \Omega | T\{A(y-a) B(x-a)\} | \Omega \rangle$$

for any $a^\mu \in \mathbb{R}^{1,3}$.

This means that

$$(\ast) = (2\pi)^4 \delta^4(p-q) \int d^4z \; e^{ip\cdot z} \langle \Omega | T\{A(z) B(0)\} | \Omega \rangle .$$

How does $\int d^4z \; e^{ip\cdot z} \langle \Omega | T\{A(z) B(0)\} | \Omega \rangle$ depend on p^μ?

Singularity is only from integral over infinite regions.

From $z^0 \in (\text{positive}, +\infty)$ region.

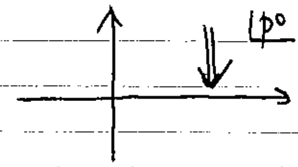
$$\int d^4z \; e^{ip\cdot z} \int dX \int \frac{d^3\vec{k}_X}{(2\pi)^3} \frac{1}{2E_X(\vec{k})} \langle \Omega | A(z) | X(\vec{k}) \rangle \langle X(\vec{k}) | B(0) | \Omega \rangle .$$

$$= \int d^4z \int dX \int \frac{d^3\vec{k}_X}{(2\pi)^3} \frac{1}{2E_X(\vec{k})} e^{ip\cdot z} e^{+i\vec{k}\cdot\vec{z}} e^{-iE_X(\vec{k})z^0} \langle \Omega | A(0) | X(\vec{k}) \rangle \langle X(\vec{k}) | B(0) | \Omega \rangle$$

$$= \int_{\text{pos.}}^{+\infty} dz^0 \int dX \frac{1}{2E_X(\vec{p})} e^{i(p^0 - E_X(\vec{p}))z^0} \langle \Omega | A(0) | X(\vec{p}) \rangle \langle X(\vec{p}) | B(0) | \Omega \rangle .$$

$$\sim \int dX \frac{i}{(p^0 - E_X(\vec{p}))} \frac{1}{2E_X(\vec{p})} \langle \Omega | A(0) | X(\vec{p}) \rangle \langle X(\vec{p}) | B(0) | \Omega \rangle .$$

The z^0 integral to $(+\infty)$ is convergent if p^0 has a positive imaginary part

Approaching from 

we encounter singularity if $\exists X$ s.t. $\langle \Omega | A | X \rangle \langle X | B | \Omega \rangle \neq 0$.

Similarly from $z^0 \in (-\infty, \text{negative})$ region,

$$\int d^4z \; e^{ip\cdot z} \int dX \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2E_X(\vec{k})} \langle \Omega | B(0) | X(\vec{k}) \rangle \langle X(\vec{k}) | A(z) | \Omega \rangle$$

$$\sim \int dX \frac{i}{(p^0 + E_X(-\vec{p}))(-2E_X(-\vec{p}))} \langle \Omega | B(0) | X(-\vec{p}) \rangle \langle X(-\vec{p}) | A(0) | \Omega \rangle .$$

(approaching from lower half plane)

In the Källen-Lehmann spectral representation

$$\int d^4z \; e^{ip\cdot z} \langle \Omega | T\{A(z)\, B(0)\} | \Omega \rangle . \sim \int dX \frac{i}{(p^2 - m_X^2 + i\varepsilon)} \rho(m_X) ,$$

particles are characterized as isolated poles $\cdot \sum_n \frac{Z_n}{p^2 - m_n^2 + i\varepsilon}$.

Multiparticle states $\Rightarrow$ branch cuts.

One particle state is contained with the normalization $|n, \vec{p}\rangle \sqrt{Z_n}$

($A/\sqrt{Z_n} |\Omega\rangle$, $B/\sqrt{Z_n} |\Omega\rangle$ contains a properly normalized 1-particle state.)

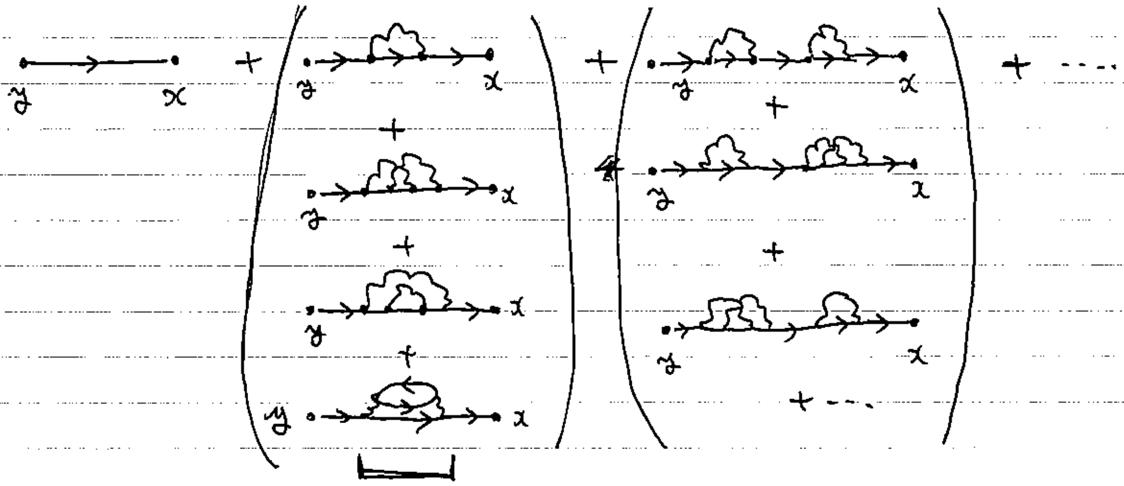
In a perturbative theory (model), like QED,

$$\int \langle 0| T\left\{ \psi_I(x)\, \bar{\psi}_I(y) \times \exp\left[-i\int_{T_-}^{T_+} V_I(t')dt'\right]\right\}|0\rangle_{\text{connected}}\, e^{ip\cdot x} d^4x = \frac{i(\not p+m)}{p^2-m^2+i\varepsilon} + \frac{i(\not p+m)}{p^2-m^2+i\varepsilon}(-i\Sigma(p))\frac{i(\not p+m)}{p^2-m^2+i\varepsilon}$$

$$+\frac{i(\not p+m)}{p^2-m^2+i\varepsilon}(-i\Sigma)\frac{i(\not p+m)}{p^2-m^2+i\varepsilon}(-i\Sigma)\frac{i(\not p+m)}{p^2-m^2+i\varepsilon}$$

$$+\cdots$$

forms a geometric series.



$(-i\Sigma)$: collection of contributions from (all the) "1-particle irreducible" graphs.

The two point function becomes.

$$\frac{i}{\not p - m - \Sigma(p) + i\varepsilon} = \frac{iZ_\psi}{\not p - m' + i\varepsilon} + (\text{other singularity}).$$

$\boxed{\S 2.2.3}$ LSZ formula

Consider

$$(*) := \int \langle\Omega|T\{\mathcal{O}_1(x_1)\cdots\mathcal{O}_k(x_k)\,\mathcal{O}_{k+1}(x_{k+1})\cdots\mathcal{O}_n(x_n)\}|\Omega\rangle$$

$$e^{-ip_1\cdot x_1}\,e^{-ip_2\cdot x_2}\cdots e^{-ip_k\cdot x_k}\;e^{iq_{k+1}\cdot x_{k+1}}\cdots e^{iq_n\cdot x_n}\,d^4x_1\sim d^4x_n.$$

$$\left[\text{ with all } (p_{i=1\sim n})^{\mu=0} > 0.\right]$$

Focusing on fully connected graphs, we find that (*) has singularity

$$\sim\left(\frac{iZ_1}{p_1^2-m_1^2+i\varepsilon}\right)\left(\frac{iZ_2}{p_2^2-m_2^2+i\varepsilon}\right)\cdots\left(\frac{iZ_n}{p_n^2-m_n^2+i\varepsilon}\right)\times\left(\text{amputated fully connected graphs}\right).$$

$$\Longrightarrow \boxed{{}^{\text{out}}\langle n_{k+1},\vec{q}_{k+1};\cdots;n_n,\vec{q}_n\,|\,n_1,\vec{p}_1;\cdots;n_k,\vec{p}_k\rangle^{\text{in}} = \left(\prod_{i=1}^{n}\sqrt{Z_i}\right)\times\left(\begin{array}{c}\text{amputated}\\ \text{fully connected graphs}\end{array}\right)}$$

§ 2.3 Feynman rules for time-ordered product

How to compute
$$\langle 0|T\{\mathcal{O}^1_I(x_1)\,\mathcal{O}^2_I(x_2)\cdots\mathcal{O}^n_I(x_n)\exp\left[-i\int dt'\,V_I(t')\right]\}|0\rangle$$
~~etc.~~ full. conn. amp. ?
$$\langle 0|T\{\mathcal{O}^1_I(x)\,\mathcal{O}^2_I(y)\}|0\rangle_{1PI}.$$

"Elementary" fields are in the form of

$$\begin{cases}\psi_I(x)=\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(u_s(\vec p)\,a_{\vec p,s}\,\underline{\underline{e^{-ip\cdot x}}}+v_s(\vec p)\,b^\dagger_{\vec p,s}\,\underline{\underline{e^{ip\cdot x}}}\right)\\ \bar\psi_I(x)=\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(\bar u_s(\vec p)\,a^\dagger_{\vec p,s}\,\underline{\underline{e^{ip\cdot x}}}+\bar v_s(\vec p)\,b_{\vec p,s}\,\underline{\underline{e^{-ip\cdot x}}}\right)\end{cases}$$
Dirac field.

= means onshell

A complex (relativistic) boson (scalar) : just drop the 4-component spinors $u_s(\vec p)$ etc.

A real relativistic boson (scalar) : drop distinction between $a^\dagger_p$ & $b^\dagger_p$.

e.g.
$$\langle 0|T\{\psi_I(x)\bar\psi_I(y)\}|0\rangle=\int\frac{d^3p}{(2\pi)^3}\left(\frac{\Theta(x^0-y^0)}{2E_p}u_s(\vec p)\bar u_s(\vec p)\,\underline{\underline{e^{-ip\cdot(x-y)}}}+\frac{\Theta(y^0-x^0)}{2E_p}v_s(\vec p)\bar v_s(\vec p)\times\underline{\underline{e^{ip\cdot(x-y)}}}\right)$$
$$=\int\frac{d^4p}{(2\pi)^4}\frac{i(\not p+m)}{(p^2-m^2+i\varepsilon)}e^{-ip\cdot(x-y)}\quad=:D(x-y).$$

the principle : bring ann. operators to the right
creat. ops'. to the left.

ann. ops / creat. ops } become a part of [,] or { } or otherwise act on |0⟩ / ⟨0| to be $\mathbb{C}$-valued.

exploit all possible combinatorics. (contraction patterns).

Therefore.

time-ordered. correlation fcns. are given by summing up contributions (amplitudes)

from all possible. contraction patterns. (Feynman diagram).

The amplitude of a given diagram is given by a product of.

[propagator] $D(x-y)$: for . each one of contractions. ([ann. creat.], {ann, creat})

and.

[vertex] remaining coefficients $\left(\text{eg. } -i \int dt' \int d^3\vec{x} \left(\bar{\psi} \gamma^\mu e A_\mu \psi \right) \right)$

integrated over all the spacetime coordinates of V_I.

Its Fourier transform version. is often more convenient.

$$\psi(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \tilde{\psi}(p). \quad \text{then} \ldots$$

for each vertex $\int d^4x \prod_i \left(e^{-ip_i\cdot x}\right) \times \text{coeff.} = (2\pi)^4 \delta^4(\Sigma_i p_i) \times \text{coeff.}$

for each propagator $\int \int e^{-ip\cdot x} e^{-iq\cdot y} D(x-y) \, d^4x \, d^4y = (2\pi)^4 \delta^4(p+q) \times D(p).$

⇓

Instead of #(vertex) spacetime integrals, we are left with.

$$\#\left(\underset{\text{(internal lines)}}{\text{propagator}}\right) - \#(\text{vertex}) = -\chi(\text{graph}) = -h_0 + h_1 \ldots$$

$h_0 := \#(\text{connected components}) = 1$ (of fully connected)

$h_1 := \#(\text{loops})$

actually $\boxed{h_1 = \#(\text{loops})}$ momentum integrals.

$h_1 = 0$	(tree level)	leading order contributions.
$h_1 = 1$	(1-loop)	next-to-leading order contributions.
⋮	⋮	⋮