§2.3. Feynman rules for time-ordered product

How to compute $\begin{cases} \langle 0 | T\{ \mathcal{O}^1_I(x_1)\, \mathcal{O}^2_I(x_2) \cdots \mathcal{O}^n_I(x_n) \exp\left[-i\int dt' V_I(t')\right]\} | 0 \rangle \\ \langle 0 | T\{ \mathcal{O}^1_I(x)\, \mathcal{O}^2_I(y)\} | 0 \rangle_{1PI} \end{cases}$ ~~etc.~~ full.conn. amp. ?

"Elementary" fields are in the form of

$$\begin{cases} \psi_I(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(u_s(\vec{p})\, a_{\vec{p}.s}\, \underline{\underline{e}}^{-ip\cdot x} + v_s(\vec{p})\, b^\dagger_{\vec{p}.s}\, \underline{\underline{e}}^{ip\cdot x} \right) \\ \bar{\psi}_I(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(\bar{u}_s(\vec{p})\, a^\dagger_{\vec{p}.s}\, \underline{\underline{e}}^{ip\cdot x} + \bar{v}_s(\vec{p})\, b_{\vec{p}.s}\, \underline{\underline{e}}^{-ip\cdot x} \right) \end{cases} \quad \text{Dirac field.}$$

$=$ means onshell

A complex (relativistic) (scalar) boson: just drop the 4-component spinors $u_s(\vec{p})$ etc.

A real relativistic (scalar) boson: drop distinction between $a^\dagger_{\vec{p}}$ & $b^\dagger_{\vec{p}}$.

e.g.

$$\langle 0 | T\{ \psi_I(x) \bar{\psi}_I(y)\} | 0 \rangle = \int \frac{d^3\vec{p}}{(2\pi)^3} \left(\frac{\Theta(x^0-y^0)}{2E_{\vec{p}}} u_s(\vec{p})\bar{u}_s(\vec{p})\, \underline{\underline{e}}^{-ip\cdot(x-y)} + \frac{\Theta(y^0-x^0)}{2E_{\vec{p}}} v_s(\vec{p})\bar{v}_s(\vec{p}) \times \underline{\underline{e}}^{ip\cdot(x-y)} \right)$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p}+m)}{(p^2-m^2+i\varepsilon)} e^{-ip\cdot(x-y)} =: D(x-y).$$

the principle: bring ann. operators to the right, creat. ops'. to the left.

ann. ops, creat. ops: become a part of $[\,,\,]$ or $\{\,\}$ or otherwise act on $|0\rangle$, $\langle 0|$ to be $\mathbb{C}$-valued.

exploit all possible combinatorics. (contraction patterns).

Supplementary notes on Dirac fermion

We use $\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi$ as the Lagrangian.

The gamma matrices satisfy the relation $\{\gamma^\mu, \gamma^\nu\} = 2\underbrace{\eta^{\mu\nu}}_{\mathrm{diag}(+,-,-,-)}$.

To the equation of motion $(i\gamma^\mu D_\mu - m)\psi = 0$, there are 2 solutions of the form $\psi(x) = u_s(\vec{p})\, \underline{e^{-ip\cdot x}} = u_s(\vec{p})\, e^{-iE_p t} e^{i\vec{p}\cdot\vec{x}}$, $(s=1.2)$.

In a chiral basis, where $\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_{2\times2} \\ \mathbb{1}_{2\times2} & 0 \end{pmatrix}$, $\gamma^i = \begin{pmatrix} 0 & \tau^i \\ -\tau^i & 0 \end{pmatrix}$,

$$u_s(\vec{p}) = \begin{pmatrix} (\sqrt{p\cdot\sigma})\xi_s \\ (\sqrt{p\cdot\bar{\sigma}})\xi_s \end{pmatrix}. \qquad \begin{pmatrix} (\sigma^\mu)_{2\times2} = (\mathbb{1}, \vec{\tau}) \\ (\bar{\sigma}^\mu)_{2\times2} = (\mathbb{1}, -\vec{\tau}) \end{pmatrix}$$

Two other solutions to the equation of motion are given by

$$\psi(x) = v_s(\vec{p})\, \underline{e^{ip\cdot x}} = v_s(\vec{p})\, e^{iE_p t} e^{-i\vec{p}\cdot\vec{x}} \qquad (s=1.2).,$$

with $v_s(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi_s \\ -\sqrt{p\cdot\bar{\sigma}}\,\xi_s \end{pmatrix}$ in the chiral basis above.

We usually take the 2-component spinors ξ_r to be orthonormal, $\xi_r^\dagger \xi_s = \delta_{rs}$.

Q One can verify that the following relations hold: [see Peskin-Schröeder §3]

$$\boxed{\begin{aligned} &\sum_{s=1.2} u_s(\vec{p})\bar{u}_s(\vec{p}) = \not{p} + m, \qquad \sum_{s=1.2} v_s(\vec{p})\bar{v}_s(\vec{p}) = \not{p} - m, \\ &u_r^\dagger(\vec{p}) u_s(\vec{p}) = 2E_p \xi_r^\dagger \xi_s = v_r^\dagger(\vec{p}) v_s(\vec{p}) \\ &\bar{u}_r(\vec{p}) u_s(\vec{p}) = 2m\, \xi_r^\dagger \xi_s = -\bar{v}_r(\vec{p}) v_s(\vec{p}). \end{aligned}}$$

In the non-relativistic limit, $\frac{1}{\sqrt{2E_p}} u_s(\vec{p})$ becomes $\frac{1}{\sqrt{2}}\begin{pmatrix} \xi_s \\ \xi_s \end{pmatrix}$ in the chiral basis.

[$\begin{pmatrix} \xi_s \\ 0 \end{pmatrix}$. in the non-rela. basis]

Therefore.

time-ordered correlation fcns. are given by summing up contributions (amplitudes)

from all possible contraction patterns.

(Feynman diagram).

The amplitude of a given diagram is given by a product of.

[propagator] $D(x-y)$ for each one of contractions.
$\left([\text{ann. creat.}], \{\text{ann}, \text{creat}\} \right)$

and.

[vertex] remaining coefficients $\left(\text{eg. } -i \int dt' \int d^3x \left(\bar{\psi} \gamma^\mu e A_\mu \psi \right) \right)$

integrated over all the spacetime coordinates of V_I.

Its Fourier transform version is often more convenient.

$\psi(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \tilde{\psi}(p)$. then...

for each vertex $\int d^4x \prod_i \left(e^{-ip_i \cdot x} \right) \times \text{coeff.} = (2\pi)^4 \delta^4(\Sigma_i p_i) \times \text{coeff.}$

for each propagator $\iint e^{-ip\cdot x} e^{-iq\cdot y} D(x-y)\, d^4x\, d^4y = (2\pi)^4 \delta^4(p+q) \times D(p).$

$\Downarrow$

Instead of #(vertex) spacetime integrals, we are left with.

$\#\binom{\text{propagator}}{\text{internal lines}} - \#(\text{vertex}) = -\chi(\text{graph}) = -h_0 + h_1 \cdots$

$h_0 := \#(\text{connected components}) = 1$ (of fully connected)

$h_1 := \#(\text{loops})$

actually $\boxed{h_1 = \#(\text{loops})}$ momentum integrals.

$h_1 = 0$ (tree level) leading order contributions.

$h_1 = 1$ (1-loop) next-to-leading order contributions.

⋮ ⋮ ⋮

§2.4. Other quantities of interest in QFT

Time-ordered product correlation functions
(and scattering amplitudes derived from them)
are not the only class of observables in QFT's.

Two other classes of interest.

- In-In formalism / real time formalism in thermal field theory/
Schwinger-Keldysh formalism (known in many different names).

examples: $\langle \text{init. state} | T\{\mathcal{O}_1(x_1)\, \mathcal{O}_2(x_2) \cdots \mathcal{O}_n(x_n)\} | \text{init. state} \rangle$

The initial state $|\text{init. state}\rangle$ is often the ground state or thermal ensemble, and interactions (backgrounds) are turned on later.

✓ conductance (current-current correlation) in a time-dep. background field

✓ decay / scattering of particles in a thermal plasma.

✓ growth (time evolution) of inflation field fluctuations in the background of metric of expanding universe.

The time coordinate starts at $t=t_*$ for the initial state. It then goes up (operators are sorted out that way), and the comes back to t_* in this class of observables.

- Out-of-time-ordered correlation functions.

example $\langle \Omega | \left| [A(t,\vec{x}), B(0,\vec{y})] \right|^2 | \Omega \rangle$ —(*). vanishes @ $t=0$ ($\vec{x} \neq \vec{y}$). How fast does it grow?

If A and B are unitary operators

$$(*) = 2 - \left(\langle \Omega | B^\dagger A^\dagger B A | \Omega \rangle + \text{c.c.} \right).$$

There are experimental ways to measure such (*)'s.

§3 Scattering Processes at the Leading Order

§3.1 Vector field propagators.

gauge symmetry = redundant description.

$A_\mu(x,t)$ and $A'_\mu := A_\mu - (\partial_\mu \chi(x,t))$ are physically equivalent.

Choose a gauge

example: Coulomb gauge $A_0 = 0$ and $\vec{\nabla}\cdot\vec{A} = 0$.

when the photon is not coupled to matter.

Mode decomposition + canonical quantization.

$$\Rightarrow \begin{cases} \vec{A}(\vec{x},t) = \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}} \sum_{r=1,2} \left(\vec{\varepsilon}_r(\vec{p})\, a_{\vec{p},r}\, e^{-ip\cdot x} + \vec{\varepsilon}_r(\vec{p})\, a^\dagger_{\vec{p},r}\, e^{ip\cdot x} \right) \\ A_0(\vec{x},t) = 0. \end{cases}$$

$\vec{p}\cdot\vec{\varepsilon}_r(\vec{p}) = 0$. (transverse polarization)

This results in

$$G_{ij}(x,y) := \langle 0 | T\{A_i(x) A_j(y)\} | 0 \rangle$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{i\, e^{-ip\cdot(x-y)}}{p^2+i\varepsilon} \left[\sum_r \vec{\varepsilon}_r(\vec{p}) \otimes \vec{\varepsilon}_r(\vec{p})\right]_{ij}. \qquad \left[\mathbb{1}_{ij} - \frac{\vec{p}_i \otimes \vec{p}_j}{|\vec{p}|^2}\right].$$

But this Green function does not satisfy the EoM.

$$\left[\partial^2 \delta_\mu{}^\kappa - \partial_\mu \partial^\kappa\right] G_{\kappa\lambda} J^\lambda = J_\mu \ldots \text{———} (※)$$

Modification A (temporal / axial gauge)

$$\boxed{G_{\mu\nu} = \frac{-i}{p^2+i\varepsilon}\left[\eta_{\mu\nu} - \frac{(p_\mu n_\nu + n_\mu p_\nu)}{(p\cdot n)} + \frac{p_\mu p_\nu\, n^2}{(p\cdot n)^2}\right]}$$ for some n_μ.

For $n^\mu = (1,\vec{0})$ $\quad n^\mu G_{\mu\nu} = 0$, $\quad p^i G_{i\nu} J^\nu = \frac{i\rho(p)}{p^0}$, (※)

are all satisfied. [use $\partial^\mu J_\mu = 0$]

The 3×3 part is $\left[-\mathbb{1} + \frac{\vec{p}\otimes\vec{p}}{(p^0)^2}\right]$

(Modification B (Coulomb gauge)

replace by $\left[\eta_{\mu\nu} - \frac{(p_\mu n_\nu + n_\mu p_\nu)(n\cdot p)}{(p\cdot n)^2 - p^2 n^2} + \frac{p_\mu p_\nu\, n^2}{(p\cdot n)^2 - p^2 n^2}\right]$)

The Coulomb gauge (or axial gauge, temporal gauge) propagator is rarely use for practical computations in QFT.

More convenient choice is

$$\boxed{\Theta_{\mu\nu} = \int \frac{d^4p}{(2\pi)^4} \frac{i e^{-ip\cdot(x-y)}}{p^2+i\varepsilon} \left[\eta_{\mu\nu} + (\xi-1)\frac{p_\mu p_\nu}{p^2}\right].}$$

- $\xi = 1$: Feynman gauge.
- $\xi = 0$: Landau gauge
- $\xi = \infty$: unitary gauge.

(For derivation, see Peskin-Schroeder §9.).

§3.2. $e^+ + e^- \to \mu^+ + \mu^-$ cross section

We use a Lagrangian $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi$.

$$D_\mu = (\partial_\mu - ieQA_\mu)$$

$$\left(\begin{array}{l} e>0 \\ Q : \text{charge} \quad (Q_e = -1) \end{array}\right)$$

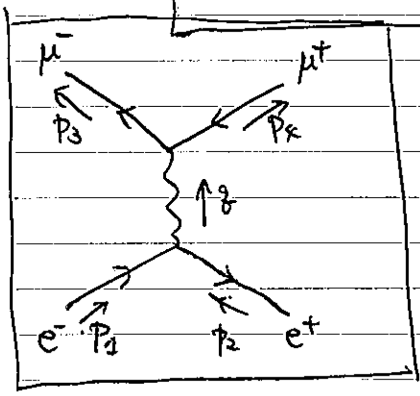
Consider.

$$(*) = \langle 0 | T\Big\{ \psi_{m.c}(-p_3)\, \bar{\psi}_{m.d}(-p_4)\, \psi_{e.b}(p_2)\, \bar{\psi}_{e.a}(p_1) \int d^4x\, ie\{Q_m(\bar{\psi}_m\gamma^\mu\psi_m) + Q_e(\bar{\psi}_e\gamma^\mu\psi_e)\}A_\mu$$
$$\int d^4y\, ie\{Q_m(\bar{\psi}_m\gamma^\nu\psi_m) + Q_e(\bar{\psi}_e\gamma^\nu\psi_e)\}A_\nu \times \frac{1}{2!}\Big\}|0\rangle .$$

$(p_1^{\mu=0},\ p_2^{\mu=0},\ p_3^{\mu=0},\ p_4^{\mu=0} > 0.)$

$\sqcap$ (contraction) : pairing of annihilation and creation operators

$$\Rightarrow \frac{2}{2}\times \frac{i(\not{p}_3+m_m)_{cf}}{p_3^2-m_m^2+i\varepsilon}\, ieQ_m\,[\gamma^\mu]_{fg}\, \frac{i(-\not{p}_4+m_m)_{gd}}{p_4^2-m_m^2+i\varepsilon} \times\left(\frac{-i\eta_{\mu\nu}}{q^2+i\varepsilon}\right)$$
$$\times \frac{i(-\not{p}_2+m_e)_{bh}}{p_2^2-m_e^2+i\varepsilon}\, ieQ_e\,[\gamma^\nu]_{hj}\, \frac{i(\not{p}_1+m_e)_{ja}}{p_1^2-m_e^2+i\varepsilon}$$
$$\times (2\pi)^4\delta^4(p_1+p_2-q) \times (2\pi)^4\delta^4(q-p_3-p_4)\ \frac{d^4q}{(2\pi)^4}.$$



$$\mathrm{Res}((*)) = \left[\bar{u}_r(\vec{p}_3)\gamma^\mu v_s(\vec{p}_4)\right]\left[\bar{v}_{s'}(\vec{p}_2)\gamma^\nu u_{r'}(\vec{p}_1)\right]\frac{\eta_{\mu\nu}}{(p_1+p_2)^2+i\varepsilon}$$
$$\times\ i(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)\times(e^2Q_eQ_\mu)$$
$$= (2\pi)^4\delta^4(p_1+p_2-p_3-p_4)\ i\mathcal{M}(e^+_{s'}e^-_{r'} \to \mu^+_s\mu^-_r).$$

(LSZ formula)

Now, we hope to compute $|\mathcal{M}|^2$ (and then the cross section).

But the expression is a mess, when we retain m_e, m_μ etc. r, s, r', s'.

The expression of $|\mathcal{M}|^2$ simplifies when we think of

$$\longrightarrow \sum_{r,s} \left|\mathcal{M}\left(e^-_{r'}+e^+_{s'} \to \mu^-_r+\mu^+_s\right)\right|^2$$

ignore the spin of μ^+,μ^- in the final state.

$$\longrightarrow \frac{1}{2}\sum_{r'=1}^{2} \frac{1}{2}\sum_{s'=1}^{2} \sum_{r,s=1}^{2} \left|\mathcal{M}\left(e^-_{r'}+e^+_{s'} \to \mu^-_r+\mu^+_s\right)\right|^2$$

initial state e^+, e^- are not polarized.

Now

$$\overline{\Sigma|\mathcal{M}|^2} = \frac{1}{4}\sum_{r',s',r,s}\left(\frac{e^2 Q_e Q_\mu}{(p_1+p_2)^2}\right)^2 \times \left[\bar{u}_r(\vec{p}_3)\gamma^\mu v_s(\vec{p}_4)\right]\left[\bar{v}_s(\vec{p}_4)\gamma^\kappa u_r(\vec{p}_3)\right] \times \left[\bar{v}_{s'}(\vec{p}_2)\gamma_\mu u_{r'}(\vec{p}_1)\right]\left[\bar{u}_{r'}(\vec{p}_1)\gamma_\kappa v_{s'}(\vec{p}_2)\right]$$

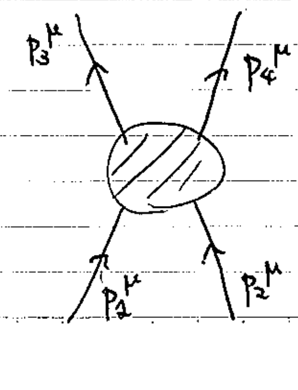
$$= \frac{1}{4}\left(\frac{e^2 Q_e Q_\mu}{(p_1+p_2)^2}\right)^2 \times \mathrm{Tr}_{4\times4}\left[\gamma^\mu(\not{p}_4-m_\mu)\gamma^\kappa(\not{p}_3+m_\mu)\right] \times \mathrm{Tr}_{4\times4}\left[\gamma_\mu(\not{p}_1+m_e)\gamma_\kappa(\not{p}_2-m_e)\right]$$

(see. suppl. notes.)

$$= \frac{1}{4}\left(\frac{e^2 Q_e Q_\mu}{(p_1+p_2)^2}\right)^2 \times \left\{-m_\mu^2\eta^{\mu\kappa} - (p_3\cdot p_4)\eta^{\mu\kappa} + p_4^\mu p_3^\kappa + p_4^\kappa p_3^\mu\right\}\times 4 \times \left\{-m_e^2\eta_{\mu\kappa} - (p_1\cdot p_2)\eta_{\mu\kappa} + p_{1\mu}p_{2\kappa} + p_{1\kappa}p_{2\mu}\right\}\times 4$$

$$= 4\left(\frac{e^2 Q_e Q_\mu}{(p_1+p_2)^2}\right)^2 \times \Big[\left\{(p_3\cdot p_4)+m_\mu^2\right\}\left\{(p_1\cdot p_2)+m_e^2\right\} 4 + \left\{(p_1\cdot p_4)(p_2\cdot p_3)+(p_4\cdot p_2)(p_1\cdot p_3)\right\}\times 2 - \left[(p_1\cdot p_2)+m_e^2\right](p_3\cdot p_4)\, 2 - \left[(p_3\cdot p_4)+m_\mu^2\right](p_1\cdot p_2)\cdot 2\Big].$$

It is conventional to use the following variables (Mandelstam variables) in 2-body $\to$ 2-body scattering:



$$s := (p_1+p_2)^2$$
$$t := (p_1-p_3)^2$$
$$u := (p_1-p_4)^2.$$

Due to the momentum conservation

$$(p_1+p_2-p_3-p_4)^\mu = 0,$$

there is a relation

$$s+t+u = m_1^2+m_2^2+m_3^2+m_4^2.$$

Using these kinematical variables, the spin-sum/average ME can be rewritten as follows:

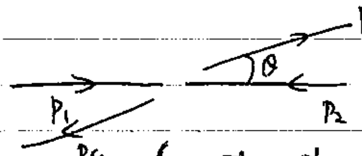
$$\frac{1}{4}\sum_{r',s'}\sum_{r,s}|\mathcal{M}|^2 = 4\left(\frac{e^2 Q_e Q_\mu}{s}\right)^2\left[s^2+\frac{1}{2}\left\{(-u+m_e^2+m_\mu^2)^2+(-t+m_e^2+m_\mu^2)^2\right\} - s\left(\frac{s-2m_\mu^2}{2}\right) - s\left(\frac{s-2m_e^2}{2}\right)\right]$$

$$= 4\left(\frac{e^2 Q_e Q_\mu}{s}\right)^2\left[\frac{u^2+t^2}{2}+(s-u-t)(m_e^2+m_\mu^2)+(m_e^2+m_\mu^2)^2\right].$$

Here are two limits of interest.

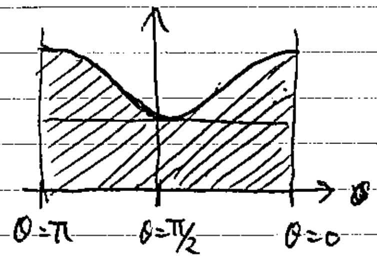
- High energy limit : $(2m_\mu)^2 \ll s$.

$$\Rightarrow \frac{1}{4}\sum_{r's'r.s}|\mathcal{M}|^2 \cong 2(e^2Q_eQ_\mu)^2\left(\frac{u^2+t^2}{s^2}\right) = (e^2Q_eQ_\mu)^2(1+\cos^2\theta)$$



(center of mass frame scattering angle)

$$\begin{cases} t \simeq -s\cdot\sin^2\left(\frac{\theta}{2}\right) \\ u \simeq -s\,\cos^2\left(\frac{\theta}{2}\right)\end{cases}$$



Using [homework I-1]

$$d\sigma \cong \frac{d(\cos\theta)}{32\pi s}(\beta\simeq 1)\left(\frac{1}{4}\sum_{r's'r.s}|\mathcal{M}|^2\right) \cong \left[d(\cos\theta)(1+\cos^2\theta) \Rightarrow \frac{8}{3}\right]\times\frac{\pi\alpha_e^2}{2s}Q_\mu^2 \Rightarrow \boxed{\frac{4\pi}{3}\frac{\alpha_e^2}{s}(Q_\mu)^2}$$

- Just above the threshold : $s\sim(2m_\mu)^2$ $\quad(\Rightarrow t\sim u\sim -m_\mu^2)$

$$\Rightarrow \frac{1}{4}\sum_{r's'r.s}|\mathcal{M}|^2 \cong 2(e^2Q_eQ_\mu)^2 \qquad (\theta\text{-indep.})$$

Using [homework I-1]

$$\int d\sigma \cong \int_{-1}^{+1} d(\cos\theta)\,\frac{\beta}{32\pi s}\left(\frac{1}{4}\sum_{r's'r.s}|\mathcal{M}|^2\right) \cong \boxed{\pi\frac{\alpha_e^2}{s}(Q_\mu)^2\times\beta}$$

$$\boxed{\beta=\frac{|\vec{p}_\mu|}{E_\mu}}\qquad \boxed{\alpha_e := \frac{e^2}{4\pi}}$$

Supplementary notes

$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$: the definition of gamma matrices.

We can use this anti-commutation relation to derive the followings.

First,

$$\mathrm{tr}_{4\times4}[\gamma^\mu\gamma^\nu] \underset{\uparrow}{=} \frac{1}{2}\left(\mathrm{tr}[\gamma^\mu\gamma^\nu] + \mathrm{tr}[\gamma^\nu\gamma^\mu]\right) = \frac{1}{2}\mathrm{tr}[\{\gamma^\mu, \gamma^\nu\}] = \frac{2}{2}\mathrm{tr}[\mathbb{1}_{4\times4}]\eta^{\mu\nu} = 4\eta^{\mu\nu}.$$

cyclic rotation within a trace.

Secondly,

$$\begin{aligned}\mathrm{tr}_{4\times4}[\gamma^\mu\gamma^\nu\gamma^\kappa\gamma^\lambda] &= \mathrm{tr}_{4\times4}[\{\gamma^\mu, \gamma^\nu\}\gamma^\kappa\gamma^\lambda] - \mathrm{tr}_{4\times4}[\gamma^\nu\gamma^\mu\gamma^\kappa\gamma^\lambda] \\ &= 2\eta^{\mu\nu}\mathrm{tr}[\gamma^\kappa\gamma^\lambda] - \mathrm{tr}[\gamma^\nu\{\gamma^\mu, \gamma^\kappa\}\gamma^\lambda] + \mathrm{tr}[\gamma^\nu\gamma^\kappa\gamma^\mu\gamma^\lambda] \\ &= 2\eta^{\mu\nu}\mathrm{tr}[\gamma^\kappa\gamma^\lambda] - 2\eta^{\mu\kappa}\mathrm{tr}[\gamma^\nu\gamma^\lambda] + \mathrm{tr}[\gamma^\nu\gamma^\kappa\{\gamma^\mu, \gamma^\lambda\}] \\ &\quad - \mathrm{tr}[\gamma^\nu\gamma^\kappa\gamma^\lambda\gamma^\mu] \\ &= 2\times4\left(\eta^{\mu\nu}\eta^{\kappa\lambda} - \eta^{\mu\kappa}\eta^{\nu\lambda} + \eta^{\mu\lambda}\eta^{\nu\kappa}\right) - \mathrm{tr}[\gamma^\mu\gamma^\nu\gamma^\kappa\gamma^\lambda].\end{aligned}$$

Therefore $\mathrm{tr}_{4\times4}[\gamma^\mu\gamma^\nu\gamma^\kappa\gamma^\lambda] = 4\left(\eta^{\mu\nu}\eta^{\kappa\lambda} - \eta^{\mu\kappa}\eta^{\nu\lambda} + \eta^{\mu\lambda}\eta^{\nu\kappa}\right)$.