§3.2. $e^+ + e^- \longrightarrow \mu^+ + \mu^-$ cross section

We use a Lagrangian $\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi$.

$$D_\mu = (\partial_\mu - ieQA_\mu)$$

$$\begin{pmatrix} e > 0. \\ Q : \text{charge}. \quad (Q_e = -1) \end{pmatrix}$$

Consider.

$$(*) = \langle 0 | T \Big\{ \psi_{m.c}(-p_3)\, \bar{\psi}_{m.d}(-p_4)\, \psi_{e.b}(p_2)\, \bar{\psi}_{e.a}(p_1) \int d^4x\; ie\left\{Q_m(\bar{\psi}_m \gamma^\mu \psi_m) + Q_e(\bar{\psi}_e\gamma^\mu\psi_e)\right\} A_\mu \int d^4y\; ie\left\{Q_m(\bar{\psi}_m\gamma^\nu\psi_m) + Q_e(\bar{\psi}_e\gamma^\nu\psi_e)\right\} A_\nu \times \frac{1}{2!} \Big\} |0\rangle .$$

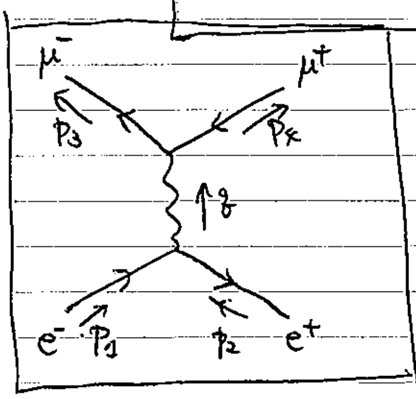
$$\left(p_1^{\mu=0},\ p_2^{\mu=0},\ p_3^{\mu=0},\ p_4^{\mu=0} > 0. \right)$$

$\sqcap$ (contraction) : pairing of annihilation and creation operators

$$\Rightarrow \frac{2}{2} \times \frac{i(\not{p}_3 + m_m)_{cf}}{p_3^2 - m_m^2 + i\varepsilon}\, ieQ_m\, [\gamma^\mu]_{fg}\, \frac{i(-\not{p}_4 + m_m)_{gd}}{p_4^2 - m_m^2 + i\varepsilon} \times \left(\frac{-i\eta_{\mu\nu}}{q^2 + i\varepsilon} \right)$$

$$\times \frac{i(-\not{p}_2 + m_e)_{bh}}{p_2^2 - m_e^2 + i\varepsilon}\, ieQ_e\, [\gamma^\nu]_{hj}\, \frac{i(\not{p}_1 + m_e)_{ja}}{p_1^2 - m_e^2 + i\varepsilon}$$

$$\times (2\pi)^4 \delta^4(p_1 + p_2 - q) \times (2\pi)^4 \delta^4(q - p_3 - p_4) \quad \frac{d^4q}{(2\pi)^4}$$



$$\mathrm{Res}\,((*)) = \left[\bar{u}_r(\vec{p}_3)\gamma^\mu v_s(\vec{p}_4)\right]\left[\bar{v}_{s'}(\vec{p}_2)\gamma^\nu u_{r'}(\vec{p}_1)\right] \frac{\eta_{\mu\nu}}{(p_1+p_2)^2 + i\varepsilon}$$

$$\times\ i(2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \times \left(e^2 Q_e Q_\mu\right)$$

$$= (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4)\; i\mathcal{M}\left(e^+_{s'} e^-_{r'} \to \mu^+_{s} \mu^-_{r}\right).$$

(LSZ formula)

Now, we hope to compute $|\mathcal{M}|^2$ (and then the cross section).

But the expression is a mess, when we retain m_e, m_μ etc. $r.s,\ r'.s'$.

§3.2.1 High-energy limit and threshold limit (unpolarized case)

The expression of $|\mathcal{M}|^2$ simplifies when we think of

$$\longrightarrow \sum_{r,s} \left| \mathcal{M}\left(e^-_{r'} + e^+_{s'} \to \mu^-_r + \mu^+_s\right)\right|^2$$

ignore the spin of μ^+, μ^- in the final state.

$$\longrightarrow \frac{1}{2}\sum_{r'=1}^{2} \frac{1}{2}\sum_{s'=1}^{2} \sum_{r,s=1}^{2} \left| \mathcal{M}\left(e^-_{r'} + e^+_{s'} \to \mu^-_r + \mu^+_s\right)\right|^2$$

initial state e^+, e^- are not polarized.

Now

$$\overline{\Sigma|\mathcal{M}|^2} = \frac{1}{4}\sum_{r',s',r,s} \left(\frac{e^2 Q_e Q_\mu}{(p_1+p_2)^2}\right)^2 \times \left[\bar{u}_r(\vec{p}_3)\gamma^\mu v_s(\vec{p}_4)\right]\left[\bar{v}_s(\vec{p}_4)\gamma^\kappa u_r(\vec{p}_3)\right]$$
$$\times \left[\bar{v}_{s'}(\vec{p}_2)\gamma_\mu u_{r'}(\vec{p}_1)\right]\left[\bar{u}_{r'}(\vec{p}_1)\gamma_\kappa v_{s'}(\vec{p}_2)\right]$$

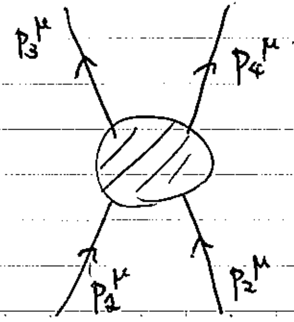
$$= \frac{1}{4}\left(\frac{e^2 Q_e Q_\mu}{(p_1+p_2)^2}\right)^2 \times \mathrm{Tr}_{4\times4}\left[\gamma^\mu(\not{p}_4 - m_\mu)\gamma^\kappa(\not{p}_3 + m_\mu)\right] \times \mathrm{Tr}_{4\times4}\left[\gamma_\mu(\not{p}_1 + m_e)\gamma_\kappa(\not{p}_2 - m_e)\right]$$

(see. suppl. notes.)

$$= \frac{1}{4}\left(\frac{e^2 Q_e Q_\mu}{(p_1+p_2)^2}\right)^2 \times \left\{-m_\mu^2\eta^{\mu\kappa} - (p_3\cdot p_4)\eta^{\mu\kappa} + p_4^\mu p_3^\kappa + p_4^\kappa p_3^\mu\right\} \times 4$$
$$\times \left\{-m_e^2\eta_{\mu\kappa} - (p_1\cdot p_2)\eta_{\mu\kappa} + p_{1\mu}p_{2\kappa} + p_{1\kappa}p_{2\mu}\right\} \times 4$$

$$= 4\left(\frac{e^2 Q_e Q_\mu}{(p_1+p_2)^2}\right)^2 \times \Big[\left\{(p_3\cdot p_4) + m_\mu^2\right\}\left\{(p_1\cdot p_2) + m_e^2\right\} 4$$
$$+ \left\{(p_1\cdot p_4)(p_2\cdot p_3) + (p_4\cdot p_2)(p_1\cdot p_3)\right\} \times 2$$
$$- \left[(p_1\cdot p_2) + m_e^2\right](p_3\cdot p_4)\, 2 - \left[(p_3\cdot p_4) + m_\mu^2\right](p_1\cdot p_2)\cdot 2\Big].$$

It is conventional to use the following variables (Mandelstam variables) in 2-body → 2-body scattering:



$$s := (p_1+p_2)^2$$
$$t := (p_1-p_3)^2$$
$$u := (p_1-p_4)^2.$$

Due to the momentum conservation

$$(p_1+p_2-p_3-p_4)^\mu = 0,$$

there is a relation

$$s+t+u = m_1^2+m_2^2+m_3^2+m_4^2.$$

(Q: verify this relation.)

Supplementary notes

$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$: the definition of gamma matrices.

We can use this anti-commutation relation to derive the followings.

First,

$$\mathrm{tr}_{4\times4}[\gamma^\mu\gamma^\nu] \underset{\uparrow}{=} \frac{1}{2}\left(\mathrm{tr}[\gamma^\mu\gamma^\nu] + \mathrm{tr}[\gamma^\nu\gamma^\mu]\right) = \frac{1}{2}\mathrm{tr}[\{\gamma^\mu, \gamma^\nu\}] = \frac{2}{2}\mathrm{tr}[\mathbb{1}_{4\times4}]\eta^{\mu\nu}$$

cyclic rotation within a trace.

$$= 4\eta^{\mu\nu}.$$

Secondly,

$$\mathrm{tr}_{4\times4}[\gamma^\mu\gamma^\nu\gamma^\kappa\gamma^\lambda] = \mathrm{tr}_{4\times4}[\{\gamma^\mu, \gamma^\nu\}\gamma^\kappa\gamma^\lambda] - \mathrm{tr}_{4\times4}[\gamma^\nu\gamma^\mu\gamma^\kappa\gamma^\lambda]$$

$$= 2\eta^{\mu\nu}\mathrm{tr}[\gamma^\kappa\gamma^\lambda] - \mathrm{tr}[\gamma^\nu\{\gamma^\mu, \gamma^\kappa\}\gamma^\lambda] + \mathrm{tr}[\gamma^\nu\gamma^\kappa\gamma^\mu\gamma^\lambda]$$

$$= 2\eta^{\mu\nu}\mathrm{tr}[\gamma^\kappa\gamma^\lambda] - 2\eta^{\mu\kappa}\mathrm{tr}[\gamma^\nu\gamma^\lambda] + \mathrm{tr}[\gamma^\nu\gamma^\kappa\{\gamma^\mu, \gamma^\lambda\}] - \mathrm{tr}[\gamma^\nu\gamma^\kappa\gamma^\lambda\gamma^\mu]$$

$$= 2\times4\left(\eta^{\mu\nu}\eta^{\kappa\lambda} - \eta^{\mu\kappa}\eta^{\nu\lambda} + \eta^{\mu\lambda}\eta^{\nu\kappa}\right) - \mathrm{tr}[\gamma^\mu\gamma^\nu\gamma^\kappa\gamma^\lambda].$$

Therefore $\mathrm{tr}_{4\times4}[\gamma^\mu\gamma^\nu\gamma^\kappa\gamma^\lambda] = 4\left(\eta^{\mu\nu}\eta^{\kappa\lambda} - \eta^{\mu\kappa}\eta^{\nu\lambda} + \eta^{\mu\lambda}\eta^{\nu\kappa}\right).$

Using these kinematical variables, the spin-sum/average ME can be rewritten as follows:

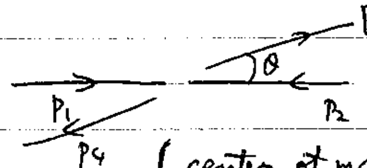
$$\frac{1}{4}\sum_{r',s'}\sum_{r,s}|\mathcal{M}|^2 = 4\left(\frac{e^2 Q_e Q_\mu}{s}\right)^2 \left[s^2 + \frac{1}{2}\left\{(-u+m_e^2+m_\mu^2)^2 + (-t+m_e^2+m_\mu^2)^2\right\} - s\left(\frac{s-2m_\mu^2}{2}\right) - s\left(\frac{s-2m_e^2}{2}\right) \right]$$

$$= 4\left(\frac{e^2 Q_e Q_\mu}{s}\right)^2 \left[\frac{u^2+t^2}{2} + (s-u-t)(m_e^2+m_\mu^2) + (m_e^2+m_\mu^2)^2 \right].$$

Here are two limits of interest.

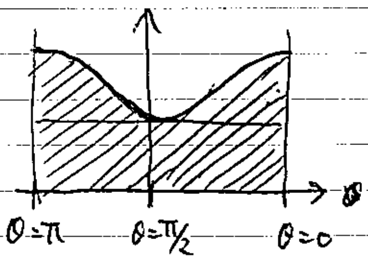
- High energy limit : $(2m_\mu)^2 \ll s$.

$$\Rightarrow \frac{1}{4}\sum_{r',s',r,s}|\mathcal{M}|^2 \cong 2(e^2 Q_e Q_\mu)^2\left(\frac{u^2+t^2}{s^2}\right) = (e^2 Q_e Q_\mu)^2(1+\cos^2\theta)$$



(center of mass frame scattering angle.)

$$\begin{cases} t \simeq -s\cdot\sin^2\left(\frac{\theta}{2}\right) \\ u \simeq -s\cos^2\left(\frac{\theta}{2}\right) \end{cases}$$



Using [homework I-1]

$$d\sigma \cong \frac{d(\cos\theta)}{32\pi s}(\beta\simeq 1)\left(\frac{1}{4}\sum_{r's'r,s}|\mathcal{M}|^2\right) \cong \left[d(\cos\theta)(1+\cos^2\theta) \Rightarrow \frac{8}{3}\right]\times\frac{\pi\alpha_e^2}{2s}Q_\mu^2 \Rightarrow \boxed{\frac{4\pi}{3}\frac{\alpha_e^2}{s}(Q_\mu)^2}$$

- Just above the threshold : $s \sim (2m_\mu)^2$ $(\Rightarrow t\sim u\sim -m_\mu^2)$

$$\Rightarrow \frac{1}{4}\sum_{r',s',r,s}|\mathcal{M}|^2 \cong 2(e^2 Q_e Q_\mu)^2 \qquad (\theta\text{-indep.})$$

Using [homework I-1]

$$\int d\sigma \cong \int_{-1}^{+1} d(\cos\theta)\,\frac{\beta}{32\pi s}\left(\frac{1}{4}\sum_{r's'r,s}|\mathcal{M}|^2\right) \cong \boxed{\pi\frac{\alpha_e^2}{s}(Q_\mu)^2\times\beta}$$

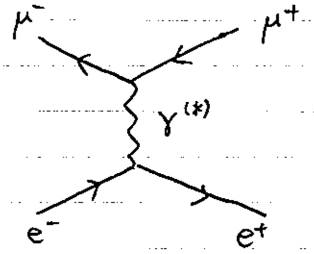
$$\boxed{\beta = \frac{|\vec{p}_\mu|}{E_\mu}}$$

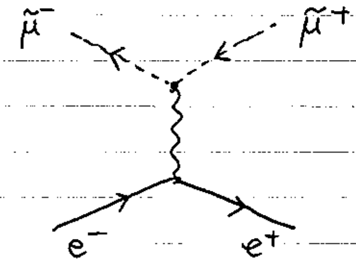
$$\boxed{\alpha_e := \frac{e^2}{4\pi}}$$

Compare

$$\left(e^-_{r'} + e^+_{s'} \to \mu^-_r + \mu^+_s \right) \quad \text{vs} \quad \left(e^-_{r'} + e^+_{s'} \longrightarrow \tilde{\mu}^- + \tilde{\mu}^+ \right)$$

spin-1/2 spin-0.





High-energy limit

$$\frac{1}{4}\sum_{r's'}\sum_{rs}|\mathcal{M}|^2 \cong \left(e^2 Q_{(\mu)} Q_{(e)}\right)^2 (1+\cos^2\theta) \quad \text{vs} \quad \frac{1}{4}\sum_{r's'}|\mathcal{M}|^2 \cong \left(e^2 Q_{(\mu)} Q_{(e)}\right)^2 \frac{(\sin^2\theta)}{2}.$$

Both are dimensionless (remember: 1st lecture of this course)

only angle dependence remains.

At the threshold

$$\frac{1}{4}\sum_{r's',rs}|\mathcal{M}|^2 \cong \left(e^2 Q_{(\mu)} Q_{(e)}\right)^2 \times 2, \qquad \frac{1}{4}\sum_{r's'}|\mathcal{M}|^2 \cong \left(e^2 Q_{(\mu)} Q_{(e)}\right)^2 \frac{\sin^2\theta}{2} \times (\beta_{(\mu)})^2$$

(see homework IV-2.)

These matrix elements are integrated

over $\frac{\beta_{(\mu)}}{32\pi s}\int_{-1}^{+1} d(\cos\theta)$

Cross sections of 2-body → 2body hard scattering processes often scale as $\sigma \sim \frac{1}{s} \sim \frac{1}{(E_{CM}/\text{GeV})^2} \times \left[\frac{(\hbar c)^2}{(\text{GeV})^2} \sim 4\times 10^{-4}\,\text{bn}\right]$

§3.2.2 Polarized case

In the high-energy limit of the $e^+e^- \to \mu^+\mu^-$ scattering, $m_e, m_\mu \ll \sqrt{s}$,
both $e^\mp$ and $\mu^\mp$ can be regarded as massless particles.

Dirac spinor for $p^\mu = (E, \vec{p}) \simeq (E, 0, 0, E)$

$$u_s(\vec{p}) \cong \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar\sigma}\,\xi \end{pmatrix} \cong \begin{pmatrix} \begin{pmatrix} 0 & \\ & \sqrt{2E} \end{pmatrix}\xi \\ \begin{pmatrix} \sqrt{2E} & \\ & 0 \end{pmatrix}\xi \end{pmatrix}$$

simplifies.

$$v_s(\vec{p}) \cong \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ -\sqrt{p\cdot\bar\sigma}\,\xi \end{pmatrix} \cong \begin{pmatrix} \begin{pmatrix} 0 & \\ & \sqrt{2E} \end{pmatrix}\xi \\ \begin{pmatrix} -\sqrt{2E} & \\ & 0 \end{pmatrix}\xi \end{pmatrix}$$

$$\left(\gamma^0 = \begin{pmatrix} & \mathbb{1} \\ \mathbb{1} & \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \tau^i \\ -\tau^i & 0 \end{pmatrix} \text{ basis}\right)$$

Then

$$\bar{v}_{s'}(\vec{p})\gamma^\nu u_{r'}(\vec{p}) = \sqrt{2E_{\vec p}}\left(0,\ \xi_{s'}^\dagger \begin{pmatrix} 0 & \\ & -1 \end{pmatrix}\vec{\tau}\begin{pmatrix} 1 & \\ & 0 \end{pmatrix}\xi_{r'} + \xi_{s'}^\dagger\begin{pmatrix} 1 & \\ & 0 \end{pmatrix}(-\vec{\tau})\begin{pmatrix} 0 & \\ & 1 \end{pmatrix}\xi_{r'}\right),$$

$$\bar{u}_r(\vec{p})\gamma^\mu v_s(-\vec{p}) = \sqrt{2E_{\vec p}}\left(0,\ \xi_r^\dagger\begin{pmatrix} 1 & \\ & 0 \end{pmatrix}\vec{\tau}\begin{pmatrix} 0 & \\ & -1 \end{pmatrix}\xi_s + \xi_r^\dagger\begin{pmatrix} 0 & \\ & 1 \end{pmatrix}(-\vec{\tau})\begin{pmatrix} 1 & \\ & 0 \end{pmatrix}\xi_s\right),$$

when $\vec{p} \parallel \hat{e}_z$ (z-axis positive direction).

$$\Rightarrow (i\mathcal{M}) = \frac{i(Q_\mu Q_e e^2)}{s+i\varepsilon}\, s \times \mathop{\mathrm{tr}}_{2\times2}\left[\left(\vec{\tau} - \vec{n}_{in}(n_{in}\cdot\tau)\right)\left(\xi_{r'}\otimes\xi_{s'}^\dagger\right)\right] \times \mathop{\mathrm{tr}}_{2\times2}\left[\left(\vec{\tau} - \vec{n}_{out}(n_{out}\cdot\tau)\right)\left(\xi_s\otimes\xi_r^\dagger\right)\right].$$

$\boxed{e^-_{r'=\uparrow}\ e^+_{s'=\downarrow}} \Rightarrow \mathrm{tr}\left[\left(\vec{\tau} - \hat{e}_z(e_z\cdot\tau)\right)\left(\begin{pmatrix}1\\0\end{pmatrix}\otimes(0,1)\right)\right] = (1, i, 0) \Leftarrow \boxed{\mu^-_{r=\downarrow}\ \mu^+_{s=\uparrow}}$

$\boxed{e^-_{r'=\downarrow}\ e^+_{s'=\uparrow}} \Rightarrow \mathrm{tr}\left[\left(\vec{\tau} - \hat{e}_z(e_z\cdot\tau)\right)\left(\begin{pmatrix}0\\1\end{pmatrix}\otimes(1,0)\right)\right] = (1, -i, 0) \Leftarrow \boxed{\mu^-_{r=\uparrow}\ \mu^+_{s=\downarrow}}$

$\boxed{e^-_{r'=\uparrow}, e^+_{s'=\uparrow}}$, $\boxed{e^-_{r'=\downarrow}, e^+_{s'=\downarrow}}$ $\Rightarrow (0, 0, 0) \Leftarrow \boxed{\mu^-_\uparrow\ \mu^+_\uparrow}\ \boxed{\mu^-_\downarrow\ \mu^+_\downarrow}$

When μ^- is moving out in the $(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) = \vec{n}_{out}$ direction,

multiply
$$\begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\theta & & \sin\theta \\ & 1 & \\ -\sin\theta & & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\chi & -\sin\chi & 0 \\ \sin\chi & \cos\chi & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ on } \begin{pmatrix} 1 \\ \mp i \\ 0 \end{pmatrix}$$

to obtain $(\cos\theta\cos\varphi \pm i\sin\varphi,\ \cos\theta\sin\varphi \mp i\cos\varphi,\ -\sin\theta) \times e^{\pm i\chi}$.

$$\text{So, } \mathrm{tr}_{2\times2}[\quad] \times \mathrm{tr}_{2\times2}[\quad] = \begin{cases} e^{i\varphi}e^{i\chi}(\cos\theta + 1) \\ e^{i\varphi}e^{-i\chi}(\cos\theta - 1) \\ e^{-i\varphi}e^{i\chi}(\cos\theta - 1) \\ e^{-i\varphi}e^{-i\chi}(\cos\theta + 1) \end{cases} = \begin{cases} e^{i(\varphi+\chi)}\,2\cos^2(\theta/2) \\ e^{i(\varphi-\chi)}(-2)\sin^2(\theta/2) \\ e^{i(-\varphi+\chi)}(-2)\sin^2(\theta/2) \\ e^{-i(\varphi+\chi)}\,2\cos^2(\theta/2) \end{cases} \begin{array}{l} \uparrow\downarrow \Rightarrow \uparrow\downarrow \\ \uparrow\downarrow \Rightarrow \downarrow\uparrow \\ \downarrow\uparrow \Rightarrow \uparrow\downarrow \\ \downarrow\uparrow \Rightarrow \downarrow\uparrow \\ e^-e^+ \quad \mu^-\mu^+ \\ (\vec{n}_{in}) \quad (\vec{n}_{out}) \end{array}$$

Therefore, the amplitudes for spin-polarized scatterings become

$$\left.\begin{array}{l} |\mathcal{M}(e^-_\uparrow e^+_\downarrow \longrightarrow \mu^-_\uparrow \mu^+_\downarrow)|^2 \\ |\mathcal{M}(e^-_\uparrow e^+_\downarrow \longrightarrow \mu^-_\downarrow \mu^+_\uparrow)|^2 \\ |\mathcal{M}(e^-_\downarrow e^+_\uparrow \longrightarrow \mu^-_\uparrow \mu^+_\downarrow)|^2 \\ |\mathcal{M}(e^-_\downarrow e^+_\uparrow \longrightarrow \mu^-_\downarrow \mu^+_\uparrow)|^2 \end{array}\right\} = \left\{\begin{array}{c} 4\cos^4(\theta/2) \\ 4\sin^4(\theta/2) \\ 4\sin^4(\theta/2) \\ 4\cos^4(\theta/2) \end{array}\right\} \times \left(e^2 Q(\mu) Q(e)\right)^2 .$$

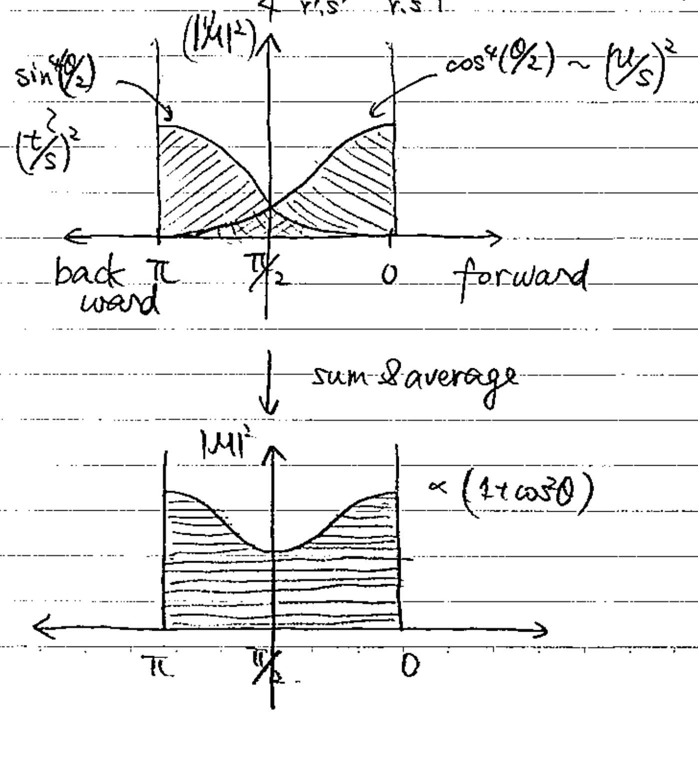
By summing up the spin in the final states,

$$\left.\begin{array}{l} \sum_{r,s} |\mathcal{M}(e^-_\uparrow e^+_\downarrow \longrightarrow \mu^-_r \mu^+_s)|^2 \\ \sum_{r,s} |\mathcal{M}(e^-_\uparrow e^+_\uparrow \longrightarrow \mu^-_r \mu^+_s)|^2 \\ \sum_{r,s} |\mathcal{M}(e^-_\downarrow e^+_\uparrow \longrightarrow \mu^-_r \mu^+_s)|^2 \\ \sum_{r,s} |\mathcal{M}(e^-_\downarrow e^+_\downarrow \longrightarrow \mu^-_r \mu^+_s)|^2 \end{array}\right\} = \left\{\begin{array}{c} 2(1+\cos^2\theta) \\ 0 \\ 2(1+\cos^2\theta) \\ 0 \end{array}\right\} \times \left(e^2 Q(\mu) Q(e)\right)^2 .$$

By taking average over the spin in the initial states,

$$\frac{1}{4}\sum_{r',s'}\sum_{r,s} |\mathcal{M}(e^-_{r'} e^+_{s'} \to \mu^-_r \mu^+_s)|^2 = (1+\cos^2\theta)\times\left(e^2 Q(\mu) Q(e)\right)^2$$

is reproduced.



$$e^-_\uparrow e^+_\downarrow \Rightarrow (\gamma^* \text{ with } j_z=+1) \Leftarrow \left\{\begin{array}{l} \mu^-_\uparrow \mu^+_\downarrow : j_{\hat{p}_3} = +1 \\ \mu^-_\downarrow \mu^+_\uparrow : j_{\hat{p}_3} = -1 \end{array}\right\}$$

$$e^-_\downarrow e^+_\uparrow \Rightarrow (\gamma^* \text{ with } j_z = -1) \Leftarrow$$

cf. see homework D-2 and/or E-1 for a much easier (and theoretically interesting) method to compute scattering amplitudes of polarized <u>massless</u> particles supplementary notes.

Maxwell equation

$$\begin{cases} \mathrm{div}\,\vec{E} = \rho/\varepsilon_0 \\ \mathrm{rot}\,\vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{j} \end{cases} \qquad \begin{cases} \mathrm{div}\,\vec{B} = 0 \\ \mathrm{rot}\,\vec{E} + \frac{\partial \vec{B}}{\partial t} = \vec{0} \end{cases}$$

$$\vec{f} = eQ(\vec{E} + \vec{v}\times\vec{B}).$$

$$\varepsilon_0\mu_0 = 1/c^2.$$

$$\varphi = \frac{1}{4\pi\varepsilon_0}\frac{eQ}{r}.$$

Physics always comes in a combination of (charge) × (field).

So it is OK. to rescale $\left[\begin{array}{l} (\text{charge}) \to (\text{charge})' = (\text{charge}) \times \lambda. \\ (\text{field}) \to (\text{field})' = (\text{field}) \times \lambda^{-1}. \end{array}\right]$.

Using this rescaling, we can set.

	cgs-esu.	cgs-esu ~~[illegible]~~ rational	cgs-emu	cgs-emu rational
ε_0	$1/4\pi$	1	$1/(4\pi c^2)$	$1/c^2$
μ_0	$4\pi/c^2$	$1/c^2$	4π	1

$$(\text{charge})_{\text{in esu}} = (\text{charge})_{\text{in emu}} \times (c_{\text{value}}).$$

$$(\text{field})_{\text{in esu}} = (\text{field})_{\text{in emu}} / (c_{\text{value}})$$

cgs-Gauss. unit system

$$A^\mu = (\varphi_{esu},\ c\cdot\vec{A}_{esu}) = (\tfrac{1}{c}\varphi_{emu},\ \vec{A}_{emu}) \quad \text{and} \quad J^\mu = (\rho_{esu},\ \tfrac{1}{c}\vec{j}_{esu}) = (c\,\rho_{emu},\ \vec{j}_{emu})$$

$$\boxed{\begin{aligned} \partial_\nu F^{\nu\mu} &= 4\pi J^\mu \\ A^0 &= \frac{eQ}{r} \\ \mathcal{L} &= -\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} - eA_\mu J^\mu \\ \mathcal{H}_{EM} &= \frac{1}{8\pi}(\vec{E}^2 + \vec{B}^2) \end{aligned}}$$

"rational" version

$$\boxed{\begin{aligned} \partial_\nu F^{\nu\mu} &= J^\mu. \\ A^0 &= \frac{1}{4\pi}\frac{eQ}{r} \\ \mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - eA_\mu J^\mu \\ \mathcal{H}_{EM} &= \frac{1}{2}(\vec{E}^2 + \vec{B}^2) \end{aligned}}$$

$$(\text{charge})_{\text{in non-rat}} = (\text{charge})_{\text{in rat. unit}} / \sqrt{4\pi}$$

$$(\text{field})_{\text{in non-rat}} = (\text{field})_{\text{in rat. unit}} \times \sqrt{4\pi}$$

• Fine str. constant

$$\alpha := \left(\frac{e^2}{4\pi\varepsilon_0}\right) \Rightarrow \frac{(e_{esu})^2}{4\pi\cdot(1/4\pi)} = \frac{(e_{Gauss})^2}{4\pi\cdot(1/4\pi)} = \frac{(e_{emu})^2}{4\pi\cdot(1/4\pi c^2)} = \boxed{\frac{(e_{Gauss\text{-}rat})^2}{4\pi\cdot 1}}$$

"e_{unit}" is the value of (positive) unit charge in a given unit system

$\alpha := e^2/4\pi$ in the lecture note.

• $J^\mu = \cancel{Q_i\bar{\psi}_i(i\gamma^\mu D_\mu - m_i)\psi_i}$ $\Sigma_i Q_i(\bar{\psi}_i\gamma^\mu\psi_i)$ means that $\Delta\mathcal{L} = \bar{\psi}_i\{i\gamma^\mu(\partial_\mu + ieQ_iA_\mu) - m_i\}\psi_i$.

sign.