§ 3.3 Crossing symmetry

Consider a scalar QED., based on the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + (D_\mu \Phi_i)^* (D^\mu \Phi_i) - (M_i)^2 |\Phi_i|^2 ;$$

where Φ_i is a complex scalar field.

$$D_\mu = (\partial_\mu - i e Q_i A_\mu).$$

Then

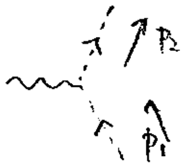
$$\mathcal{L}_{int} = \sum_i e A_\mu \cdot Q_i \left[(-i\partial^\mu \Phi_i^*)\Phi_i + \Phi_i^*(i\partial^\mu \Phi_i) \right] + e^2 A_\mu A^\mu \sum_i \left[Q_i^2 \Phi_i^* \Phi_i \right].$$

Using the fact that

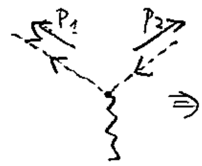
$$\left.\begin{aligned} \Phi_i(x) &= \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec p}}} \left(a_{\vec p}\, \underline{e^{-ip\cdot x}} + b_{\vec p}^\dagger\, \underline{e^{ip\cdot x}} \right) \\ \Phi_i^\dagger(x) &= \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec p}}} \left(a_{\vec p}^\dagger\, \underline{e^{ip\cdot x}} + b_{\vec p}\, \underline{e^{-ip\cdot x}} \right) \end{aligned}\right\} \Longleftarrow$$

p^μ: on the mass-shell ($p^2 = M_i^2$) with $p^{\mu=0} > 0$.

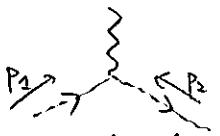
one can derive that



$\Rightarrow \; ieQ_i(p_2+p_1)^\mu$

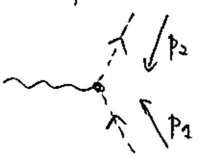


$\Rightarrow \; ieQ_i(p_1-p_2)^\mu$



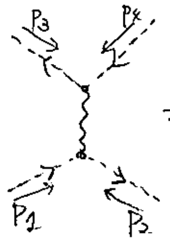
$\Rightarrow \; ieQ_i(p_1-p_2)^\mu$

All of those vertex factors are in the form of

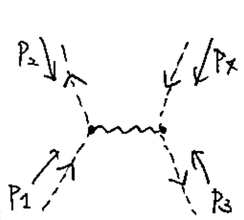


$\Rightarrow \; ieQ_i(p_1-p_2)^\mu$. when all the momenta (not necessarily $p^0>0$) are defined to be flowing <u>in</u> to the vertex.

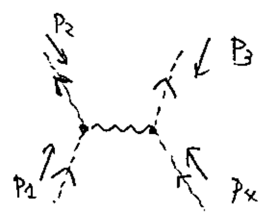
This means that



$$\Rightarrow \; i\mathcal{M} = \frac{i(eQ)^2\,(p_1-p_2)\cdot(p_4-p_3)}{(p_1+p_2)^2 + i\varepsilon} \quad \Leftarrow$$



,



$\begin{cases} \cdot\ (p_3)^0, (p_4)^0 < 0 . \\ \cdot\ (p_1+p_2)^2 > 0 \end{cases}$

$\begin{cases} \cdot\ (p_2)^0, (p_4)^0 < 0 \\ \cdot\ (p_1+p_2)^2 < 0 \end{cases}$

$\begin{cases} \cdot\ (p_2)^0, (p_3)^0 < 0 \\ \cdot\ (p_1+p_2)^2 < 0 \end{cases}$

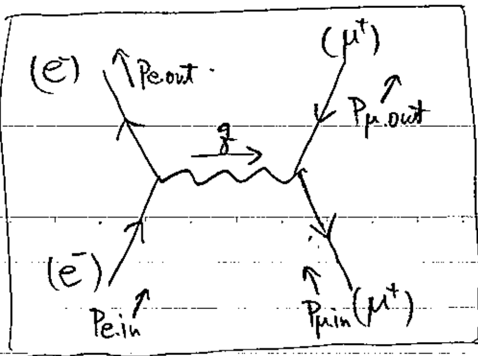
this common amplitude is used via analytic continuation at different regions of the kinematical variables.

— <u>crossing symmetry</u>

§3.4

$(e^- \mu^+ \longrightarrow e^- \mu^+)$

$\quad p_{e.in} \quad p_{\mu.in} \qquad p_{e.out} \quad p_{\mu.out}$



The time-ordered correlation fcn at $\mathcal{O}(e^2)$. [fully connected cmpn't]

is given by

$$\int \frac{i(\not{p}_{e.out}+m)}{(p_{e.out}^2-m^2+i\varepsilon)}(iQ_e e\gamma^\mu)\frac{i(\not{p}_{e.in}+m)}{(p_{e.in}^2-m^2+i\varepsilon)} \qquad \frac{-i\eta_{\mu\nu}}{(q^2+i\varepsilon)}$$

$$\frac{i(-\not{p}_{\mu.out}+M)}{(p_{\mu.out}^2-M^2+i\varepsilon)}(-iQ_\mu e\gamma^\nu)\frac{i(-\not{p}_{\mu.in}+M)}{(p_{\mu.in}^2-M^2+i\varepsilon)}$$

$$(2\pi)^4\delta^4(p_{e.out}+q-p_{e.in}) \times (2\pi)^4\delta^4(p_{\mu.out}-q-p_{\mu.in})\frac{d^4q}{(2\pi)^4}.$$

By extracting the residue, we obtain the scattering amplitude

$$\boxed{i\mathcal{M} = -i(Q_eQ_\mu e^2)\left[\bar{u}(\vec{p}_{e.out})\gamma^\mu u(\vec{p}_{e.in})\right]\left[\bar{v}(\vec{p}_{\mu.in})\gamma_\mu v(\vec{p}_{\mu.out})\right]\frac{1}{(p_{e.out}-p_{e.in})^2+i\varepsilon}}$$

After summing up the final state spins and averaging the initial state spins, ~~we have~~

$$\frac{1}{4}\sum_{r's'}\sum_{r.s}|\mathcal{M}|^2 = \frac{(Q_eQ_\mu e^2)^2}{4t^2}\mathrm{Tr}_{4\times4}\left[\gamma^\mu(\not{p}_{e.in}+m)\gamma^\kappa(\not{p}_{e.out}+m)\right]$$

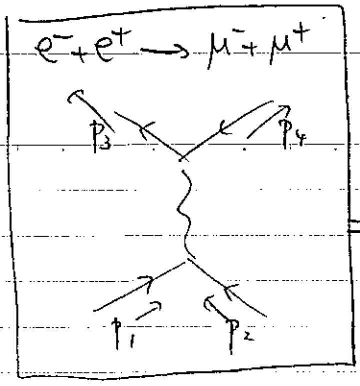
$$\mathrm{Tr}_{4\times4}\left[\gamma_\mu(\not{p}_{\mu.out}-M)\gamma_\kappa(\not{p}_{\mu.in}-M)\right]$$

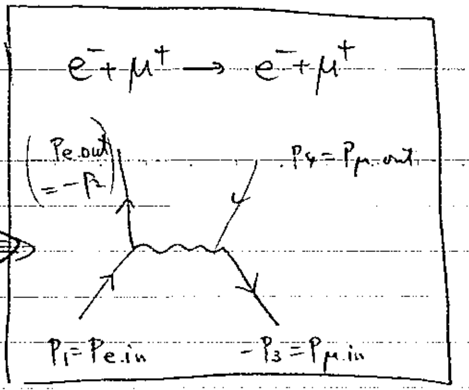
$$= \frac{(Q_\mu Q_e e^2)^2}{t^2}4\left\{m^2\eta^{\mu\kappa}-(p_{e.in}\cdot p_{e.out})\eta^{\mu\kappa}+(p_{e.in}^\mu p_{e.out}^\kappa + p_{e.out}^\mu p_{e.in}^\kappa)\right\}$$

$$\left\{M^2\eta_{\mu\kappa}-(p_{\mu.in}\cdot p_{\mu.out})\eta_{\mu\kappa}+(p_{\mu.out\,\mu}p_{\mu.in\,\kappa}+p_{\mu out\,\kappa}p_{\mu.in\,\mu})\right\}$$

$$= \frac{(Q_\mu Q_e e^2)^2}{t^2}4\left[t^2+2\left\{\left(\frac{-u+M^2+m^2}{2}\right)^2+\left(\frac{s-M^2-m^2}{2}\right)^2\right\} + t\left(-\frac{t}{2}+M^2\right)+t\left(-\frac{t}{2}+m^2\right)\right]$$

$$= \frac{(Q_\mu Q_e e^2)^2}{t^2}4\left[\frac{u^2+s^2}{2}+(t-u-s)(m^2+M^2)+(m^2+M^2)^2\right]$$





$$\boxed{\begin{aligned} s &= (p_1+p_2)^2 \Rightarrow (p_{e.in} - p_{e.out})^2 = t \\ t &= (p_3-p_1)^2 \Rightarrow (p_{e.in} + p_{\mu.in})^2 = s \\ u &= (p_4-p_1)^2 \Rightarrow (p_{\mu.out} - p_{e.in})^2 = u. \end{aligned}}$$

$$\frac{1}{4}\sum_{r',s',r,s} |\mathcal{M}|^2 \Big/ \left[(Q_\mu Q_e e^2)^2 4\right]$$

$$= \frac{1}{s^2}\left[\frac{u^2+t^2}{2} + (s-u-t)(m^2+M^2) + (m^2+M^2)\right] \Longleftrightarrow \frac{1}{t^2}\left[\frac{u^2+s^2}{2} + (t-u-s)(m^2+M^2) + (m^2+M\right.$$

for $e^- + e^+ \to \mu^- + \mu^+$ $\qquad\qquad$ $e^- + \mu^+ \to e^- + \mu^+$.

<u>crossing symmetry</u>

$\boxed{\text{Consider the } \boxed{\text{non-relativistic limit}} \text{ of the current in a scattering.}}$

$\left[\bar{u}_r(\vec{p}\,')\gamma^\mu u_s(\vec{p})\right]$ where the 4-momentua $p^\mu = (E_{\vec{p}}, \vec{p})$ and $p'^\mu = (E_{\vec{p}\,'}, \vec{p}\,')$ are both non-relativistic.

Now, in the basis where $\gamma^\mu = \begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix}$ $\quad \sigma^\mu = (\mathbb{1}, \vec{\tau})$, $\quad \bar{\sigma}^\mu = (\mathbb{1}, -\vec{\tau})$ as in P.S.

$$u_s(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi_s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi_s \end{pmatrix} \cong \begin{pmatrix} \sqrt{\left(M + \frac{\vec{p}^{\,2}}{2M} - \vec{p}\cdot\vec{\tau}\right)}\,\xi_s \\ \sqrt{\left(M + \frac{\vec{p}^{\,2}}{2M} + \vec{p}\cdot\vec{\tau}\right)}\,\xi_s \end{pmatrix}$$

$$\cong \sqrt{M}\begin{pmatrix} \left(1 - \frac{\vec{p}\cdot\vec{\tau}}{2M} + \frac{(\vec{p})^2}{8M^2}\right)\xi_s \\ \left(1 + \frac{\vec{p}\cdot\vec{\tau}}{2M} + \frac{(\vec{p})^2}{8M^2}\right)\xi_s \end{pmatrix}.$$

So,

- $$\left[\bar{u}_r(\vec{p}\,')\gamma^0 u_s(\vec{p})\right] \cong M \times \begin{pmatrix} \xi_r^\dagger \left(1 + \frac{\vec{p}\,'\cdot\vec{\tau}}{2M} + \frac{(\vec{p}\,')^2}{8M^2}\right)\cdot \mathbb{1}_{2\times2}\cdot\left(1 + \frac{\vec{p}\cdot\vec{\tau}}{2M} + \frac{(\vec{p})^2}{8M^2}\right)\xi_s \\ + \xi_r^\dagger\left(1 - \frac{\vec{p}\,'\cdot\vec{\tau}}{2M} + \frac{(\vec{p}\,')^2}{8M^2}\right)\cdot\mathbb{1}_{2\times2}\cdot\left(1 - \frac{\vec{p}\cdot\vec{\tau}}{2M} + \frac{(\vec{p})^2}{8M^2}\right)\xi_s \end{pmatrix}$$

$$\cong 2M\left\{\xi_r^\dagger\xi_s\left(1 + \frac{(\vec{p}\,')^2 + (\vec{p})^2}{8M^2}\right) + \xi_r^\dagger\frac{(\vec{p}\,'\cdot\vec{\tau})(\vec{p}\cdot\vec{\tau})}{4M^2}\xi_s\right\} + \mathcal{O}(|\vec{p}|^3)$$

$$= 2M\left\{\xi_r^\dagger\xi_s\left(1 + \frac{(\vec{p}_{av})^2}{2M^2}\right) + \frac{i(\vec{p}\,' - \vec{p})^i(\vec{p}_{av})^j}{2M^2}\overset{\varepsilon^{ijk}}{}\left(\xi_r^\dagger\frac{\vec{\tau}^k}{2}\xi_s\right)\right\} \qquad \vec{p}_{av} := \frac{\vec{p}\,' + \vec{p}}{2}$$

- $$\left[\bar{u}_r(\vec{p}\,')\gamma^i u_s(\vec{p})\right] \cong M\left\{\begin{aligned} &\xi_r^\dagger\left(1 + \frac{\vec{p}\,'\cdot\vec{\tau}}{2M} + \cdots\right)\cdot\tau^i\cdot\left(1 + \frac{\vec{p}\cdot\vec{\tau}}{2M} + \cdots\right)\xi_s \\ &- \xi_r^\dagger\left(1 - \frac{\vec{p}\,'\cdot\vec{\tau}}{2M} + \cdots\right)\cdot\tau^i\cdot\left(1 - \frac{\vec{p}\cdot\vec{\tau}}{2M} + \cdots\right)\xi_s \end{aligned}\right\}$$

$$\cong 2M\,\xi_r^\dagger\frac{(\vec{p}\,'\cdot\vec{\tau})\tau^i + \tau^i(\vec{p}\cdot\vec{\tau})}{2M}\xi_s$$

$$= 2M\left\{(\xi_r^\dagger\xi_s)\cdot\frac{p_{av}^i}{M} - i\varepsilon^{ijk}\frac{(\overset{(\vec{p}\,'-\vec{p})}{\Delta p})^j}{M}\left[\xi_r^\dagger\left(\frac{\tau^k}{2}\right)\xi_s\right]\right\}$$

⟹ In a t-channel scattering, a non-rela particle rarely changes its spin.

⟹ $[\bar{u}\gamma^\mu u]$ in the quantum scattering amplitude is approximately the classical $\boxed{2M\times u^\mu = 2\cdot p^\mu_{av}}$ apart from spin-dep. corrections suppressed by $\left(\frac{\Delta\vec{p}}{M}\right)$.

No.

Date

☆ Let us take the target (μ^+ now) mass M to be much larger than the incoming electron energy (in the CM frame).

Then $\left\{ \begin{array}{ll} [\bar{U}_r(\vec{P}_{\mu.out})\gamma^\mu U_{r'}(\vec{P}_{\mu.in})] & \text{target } \mu^- \\ [\bar{V}_{r'}(\vec{P}_{\mu.in})\gamma^\mu V_r(\vec{P}_{\mu.out})] & \text{taarget } \mu^+ \end{array} \right\}$ is approximated by

$2M_\mu \times (1.0.0.0)$,

and the photon propagator in the Feynmangauge

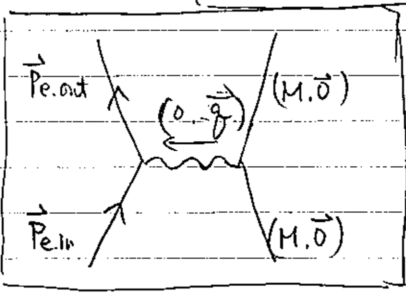
$$\left(\frac{-i\eta_{\mu\nu}}{t} = \frac{-i\eta_{\mu\nu}}{-|\vec{q}|^2}\right)$$

coupled with the current $\propto Q_\mu \cdot e \cdot (1, \vec{0})^\nu$

gives rise to the Fourier transform of the Coulomb potential

(as we are familiar with in Quantum Mechanics).

→ regardless of the spin of the target particle.



If we sum over the spin of the final state e^-, and average over the spin of the initial state e^-,

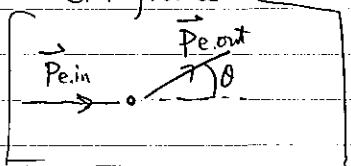
then

$$\frac{1}{2}\sum_{s,s'} |\mathcal{M}(e^-\mu^+ \to e^-\mu^+)|^2 \cong \frac{(Q_\mu Q_e e^2)^2}{(|\vec{q}|^2)^2} \frac{(2M_\mu)^2}{2} \mathrm{Tr}_{4\times4}[\gamma^0(\not{P}_{e.in}+m_e)\gamma^0(\not{P}_{e.out}+m)]$$

$$\simeq \frac{e^4}{|\vec{q}|^4}(2M_\mu)^2\, 2\left[P^0_{e.in}P^0_{e.out} + \vec{P}_{e.in}\cdot\vec{P}_{e.out} + m_e^2\right]$$

$$= \frac{e^4 \;(2M_\mu)^2}{|\vec{P}_e|^4(2\sin(\frac{\theta}{2}))^4} \cdot 4\left[E_e^2 - |P_e|^2\sin^2(\tfrac{\theta}{2})\right]$$

CM frame



$E_{e.in} \simeq E_{e.out}$

$|\vec{P}_{e.in}| \simeq |\vec{P}_{e.out}|$

(recoil of the target negligible)

The final state phase space is

$$\frac{1}{2M\cdot 2E_e\,(P_e/E_e - 0)}\int \frac{d^3\vec{P}_{e.out}}{(2\pi)^3}\frac{1}{2E_e}\frac{1}{2M_\mu}(2\pi)\,\delta(E_{e.out} - E_{e.in})$$

$$\cong \frac{d^2\Omega}{(4\pi)^2}\frac{1}{4M_\mu^2}.$$

(Rutherford scattering)

So.
$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha_e^2}{4|P_e|^4\sin^4(\theta/2)}\left[E_e^2 - P_e^2\sin^2(\theta/2)\right]} \xrightarrow{e^- \text{ non rela}} \boxed{\frac{\alpha_e^2\; m_e^2}{4|P_e|^4\sin^4(\theta/2)}}$$

What if use a polarized e^--beam?

(The target μ^+ (or ion$^+$) is not necessarily polarized.)

The e^--spin dependent term is found in

$$[\bar{u}_r \gamma^0 u_{r'}] \cong 2m_e \left[\xi_r^\dagger \xi_{r'} (1 + \cdots) + i \frac{(\vec{q} \times \vec{P}_{av})}{2m_e^2} \underbrace{\xi_r^\dagger \left(\frac{\vec{\sigma}}{2}\right) \xi_{r'}}_{\text{matrix element of the spin operator}} + \cdots \right].$$

But this term alone leads only to spin-dependent complex phase; after taking $|\mathcal{M}|^2$, it is gone.

This $\xi_r^\dagger \left(\frac{\vec{\sigma}}{2}\right) \xi_{r'}$ term : $\vec{L} \cdot \vec{S}$ coupling.

The leading order term :

$$\Delta(i\mathcal{M}) = (2m_e) \cdot (2m_{(\mu)}) (ieQ_{(e)})(-ieQ_{(\mu)}) \xi_r^\dagger \xi_{r'} \cdot \frac{-i}{-|\vec{q}|^2}$$

factor out $(2m_e)(2m_{(\mu)})$ to get the amplitude in the non-rela QM.

$$\Rightarrow \Delta(i\mathcal{M})_{\text{non-rela}}(\vec{q}) = -i \cdot \frac{(-e^2 Q_{(e)} Q_{(\mu)})}{|\vec{q}|^2} \xrightarrow[\left(\int \frac{d^3\vec{q}}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}}\right)]{\text{F.T.}} -i \frac{e^2 Q_{(e)} \cdot (-Q_{(\mu)})}{4\pi r} =: -i \underbrace{V(r)}_{\text{Coulomb potential}}$$

Now

$$\Delta(i\mathcal{M}) = (2m_e)(2m_{(\mu)}) (ieQ_{(e)})(-ieQ_{(\mu)}) \frac{-i}{-|\vec{q}|^2} \, i \frac{(\vec{q} \times \vec{P}_{av}) \cdot \langle \vec{S} \rangle}{2m_e^2}$$

$$\Rightarrow \Delta(i\mathcal{M})_{\text{non-rela}}(\vec{q}) = -i \frac{(-e^2 Q_{(e)} Q_{(\mu)})}{|\vec{q}|^2} \cdot \frac{(i\vec{q} \times \vec{P}_{av}) \cdot \langle \vec{S} \rangle}{2m_e^2} \overset{\text{F.T}}{\Rightarrow}$$

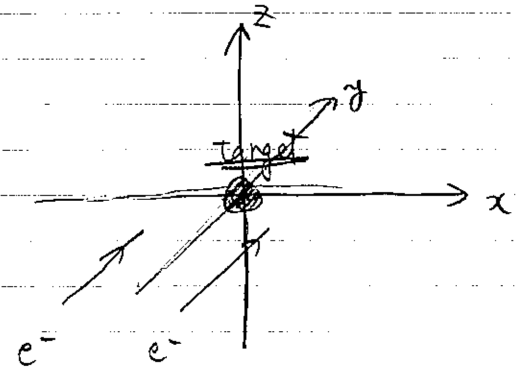
$$(\Delta V)(\vec{r}) = \frac{(\vec{\partial}_r \times \vec{P}_{av}) \cdot \langle \vec{S} \rangle}{2m_e^2} \left(\frac{e^2 Q_{(e)} \cdot (-Q_{(\mu)})}{4\pi r} \right)$$

$$= -\frac{\alpha Q_{(e)} (-Q_{(\mu)})}{2m_e^2 r^3} (\vec{r} \times \vec{P}_{av}) \cdot \langle \vec{S} \rangle$$

$$\Rightarrow + \frac{\alpha Z}{2m_e^2 r^3} \vec{L} \cdot \langle \vec{S} \rangle$$

$\left\{ \begin{array}{l} Q_{(e)} = -1 \\ -Q_{(\mu)} = \text{target charge} \Rightarrow Z \end{array} \right\}$

This $\vec{L}\cdot\vec{s}$ coupling gives rise to $\langle\vec{s}\rangle$ dependent potential.



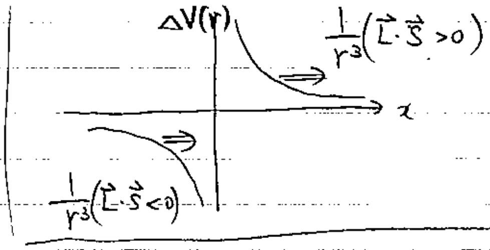
e^- approaching the target in the $0<x$ region

$\Rightarrow \vec{L}=(\vec{r}\times\vec{p}_{av})$ points to the $\hat{e}_z$ direction

e^- approaching the target in the $x<0$ region

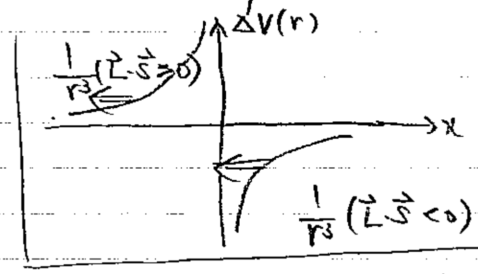
$\Rightarrow \vec{L}=(\vec{r}\times\vec{p}_{av})$ points to the $(-\hat{e}_z)$ direction.

If e^- has spin $\vec{s}\,/\!/\,\hat{e}_z$



the e^- receives a kick to the $\hat{e}_x$ direction whether it is in the $0<x$ region or in the $x<0$ region.

If e^- has spin $\vec{s}\,/\!/\,-\hat{e}_z$



the e^- receives a kick to the $-\hat{e}_x$ direction regardless of $0<x$ or $x<0$.

The asymmetry in the scattering of a polarized e^- beam in the Mott scattering is generated at 1-loop order (and beyond).

The tree-level (Born approximation) is not enough.

$\boxed{\text{Exploited in spin-resolved photoemission spectroscopy.}}$

For more about this computation, see.

"Electron Scattering without Atomic or Nuclear Excitation"

by J.W. Motz, H. Olsen and H.W. Koch

Rev. Mod. Phys. 36 (1964) 881.