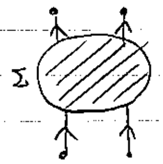
§ 4.3 BS wavefunction in a QFT process.



$$= (2\pi)^4 \delta^4(P'_{CM} - P_{CM}) \cdot G(P_{CM}; p^\mu, p'^\mu)$$

Bethe-Salpeter wavefunction

$$\boxed{G(P_{CM}; p, p') = \sum_n \chi_n(p') \frac{i}{(P_{CM}^2 - M_n^2 + i\varepsilon)} \chi_n^*(p) + \cdots}$$

$$\langle \Omega | T\{\psi_a(p_3)\, \psi_b(p_4)\} | n; \vec{P}_{CM} \rangle = (2\pi)^4 \delta^4(p_3 + p_4 - P_{CM}) \times \chi_n(p')$$

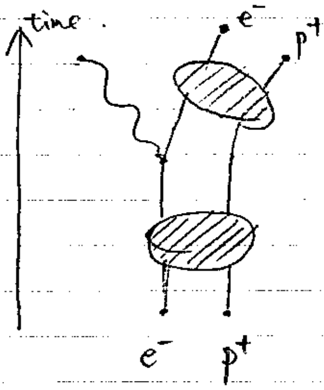
$$\chi_n(p') \cong \frac{i}{\left[\eta_a(\Delta E) + \omega' - \frac{(\eta_a \vec{P}_{CM} + \vec{p}')^2}{2m_a} + i\varepsilon\right]} \frac{i}{\left[\eta_b(\Delta E) - \omega' - \frac{(\eta_b \vec{P}_{CM} - \vec{p}')^2}{2m_b} + i\varepsilon\right]} \underline{\chi_n(\vec{p}')}$$

$$\psi_n(\vec{p}') := \int \frac{d\omega'}{2\pi} \chi_n(p') \cong \frac{i}{\left[(\Delta E) - \frac{(\vec{P}_{CM})^2}{2(m_a + m_b)} - \frac{(\vec{p}')^2}{2\mu_{ab}}\right]} \underline{\chi_n(\vec{p}')}$$

$$\psi_n(\vec{p}') \cong \sqrt{2(m_a + m_b)} \cdot \psi_n(\vec{p}')_{\text{non-rela QM}}$$

$$[-1] \quad = \quad [+1/2] \quad [-3/2]$$

Consider an atomic transition process $\boxed{(\text{Bnd state})_{②} \to (\text{Bnd state})_{①} + \gamma}$



$$i\mathcal{M} \cdot (2\pi)^4 \delta^4(p_{in}^\mu - p_{out}^\mu - k^\mu)$$

$$= \text{Residue}\left[\langle \Omega | T\{A(k)\, \psi_{e^-}(p_3)\, \psi_{p^+}(p_4)\, \psi^\dagger_{p^+}(p_2)\, \psi_{e^-}(p_1)\} | \Omega \rangle\right]$$

(LSZ formula)

kinematics

	ini. state	fin. state
		$(k, \vec{k})$ photon
e^- line	$(m_e + \eta_e(\Delta E_i) + \omega, \vec{0} + \vec{p})$	$(m_e + \eta_e(\Delta E_{f(-\vec{k})}) + \omega', -\eta_e \vec{k} + \vec{p}')$
p^+ line	$(m_p + \eta_p(\Delta E_i) - \omega, \vec{0} - \vec{p})$	$(m_p + \eta_p(\Delta E_{f(-\vec{k})}) - \omega', -\eta_p \vec{k} - \vec{p}')$

$$\boxed{\vec{p} = \vec{p}' + \eta_p \vec{k} = \vec{p}' + (1 - \eta_e)\vec{k}}$$

$$\omega = \omega' + \eta_p \{(\Delta E_i) - (\Delta E_{f(-\vec{k})})\} = \omega' + k - \eta_e \{(\Delta E_i) - (\Delta E_{f(-\vec{k})})\}$$
$$= \omega' + \eta_p k$$

$$\boxed{k = \{(\Delta E_i)_{c0} - (\Delta E_{f(-\vec{k})})_{c0}\} > 0}$$

(ΔE_f is more negative than ΔE_i)

$$i\mathcal{M}_{(\gamma\,\text{on}\,e^-\text{line})} = \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{d\omega}{2\pi}\,\chi_f^*(\omega';\vec{p}';-\vec{k})\left(\frac{i}{\left[\eta_p\Delta E_i-\omega-\frac{(\vec{p})^2}{2m_p}+i\varepsilon\right]}\right)^{-1}\chi_i(\omega,\vec{p};\vec{0})$$

$$\times\left(ieQ_e\frac{(\vec{p}_{e.in}+\vec{p}_{e.out})\cdot\vec{\varepsilon}^*(\gamma)}{2m_e}\right) \longleftarrow \mathcal{L}_{int} \ni -\left(\frac{\vec{p}\cdot(-eQ_e\vec{A})+(-eQ_e\vec{A})\cdot\vec{p}}{2m_e}\right)$$

on the e^- line.

$$= \int\frac{d^3\vec{p}}{(2\pi)^3}\,\frac{eQ_e\,\vec{\varepsilon}^*\cdot(\vec{p}_{e.in}+\vec{p}_{e.out})}{2m_e}\,\underline{\underline{\chi_f^*}}(\vec{p}';-\vec{k})\,\underline{\underline{\chi_i}}(\vec{p};\vec{0})$$

$$\times\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\,\frac{1}{\left[\eta_e(\Delta E_i)+\omega-\frac{|\vec{p}|^2}{2m_e}+i\varepsilon\right]\left[\eta_e(\Delta E_f)+\omega-k\eta_p-\frac{(\vec{p}-\vec{k})^2}{2m_e}+i\varepsilon\right]\left[\eta_p(\Delta E_i)-\omega-\frac{|\vec{p}|^2}{2m_p}+i\varepsilon\right]}$$

Noting that

$$\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\,\frac{1}{(\omega+A+i\varepsilon)(\omega+B+i\varepsilon)(-\omega+C+i\varepsilon)}$$

$$= \int\frac{d\omega}{2\pi}\,\frac{-1}{(A-B)(A+C)(B+C)}\left\{\frac{(B+C)}{(\omega+A+i\varepsilon)}-\frac{(A+C)}{(\omega+B+i\varepsilon)}+\frac{(A-B)}{(\omega-C-i\varepsilon)}\right\}$$

$$= (\log\ \text{divergence})\times\frac{\{-(B+C)+(A+C)-(A-B)=0\}}{(A-C)(A+C)(B+C)}+\frac{-\pi i}{2\pi}\,\frac{-(B+C)+(A+C)+(A-B)}{(A-B)(A+C)(B+C)}$$

$$= \frac{(-i)}{(A+C)(B+C)}$$

$$= \int\frac{d^3\vec{p}}{(2\pi)^3}\,\frac{-ieQ_e}{m_e}\,\frac{\vec{\varepsilon}^*\cdot(\vec{p}_{in}+\vec{p}_{out})}{2}\,\underline{\underline{\chi_f^*}}(\vec{p}';-\vec{k})\,\underline{\underline{\chi_i}}(\vec{p};\vec{0})\,\frac{1}{\left[(\Delta E_i)-\frac{|\vec{p}|^2}{2\mu_{ep}}\right]}\,\frac{1}{\left[(\Delta E_f(\vec{k}))-\frac{\vec{k}^2}{2(m_e+m_p)}-\frac{(\vec{p}-\eta_p\vec{k})^2}{2\mu_{ep}}\right]}$$

$$= -ieQ_e\cdot 2(m_e+m_p)\int\frac{d^3\vec{p}}{(2\pi)^3}\,\psi^*_{NR,f}(\vec{p}-\eta_p\vec{k};-\vec{k})\,\frac{\vec{\varepsilon}^*\cdot(\vec{p}+(\vec{p}-\vec{k}))}{2m_e}\,\psi_{NR,i}(\vec{p};\vec{0}).$$

$\left([i\mathcal{M}] = +1 \quad +3 \quad -\tfrac{3}{2}\times 2 \quad +0 \quad = +1.\ \text{as expected in } 1\to 2 \text{ body decay}\right)$

reproduces $\langle f|\frac{\vec{p}}{m_e}eQ|i\rangle\cdot\vec{\varepsilon} = \langle f|\dot{\vec{r}}eQ|i\rangle\cdot\vec{\varepsilon} = \langle f|-ieQ[\vec{r},H]|i\rangle\cdot\vec{\varepsilon}$

$= -i\langle f|\underline{eQ\vec{r}}|i\rangle\cdot\vec{\varepsilon}\,(\Delta E_i-\Delta E_f)$

$\leftrightarrow$ ME of electric $\underline{\text{dipole}}$

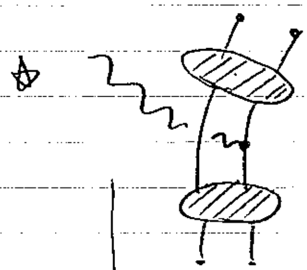
$$d\Gamma \simeq \frac{1}{2m_H}\frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2m_H}\frac{1}{2k}2\pi\,\delta(k-(\Delta E_i-\Delta E_f))\,|\mathcal{M}|^2 \simeq \frac{d\Omega}{8\pi^2}k^3\,|\langle f|\vec{d}|i\rangle|^2 \sim \alpha\cdot(m_e\alpha^2)^3\cdot\frac{1}{(m_e\alpha)^2}$$

$$\sim \mathcal{O}(m_e\alpha^5).$$

$\boxed{\text{dipole emission}}$

<u>comments</u>

☆ $\Gamma \sim O(m_e\alpha^5)$ is smaller than the fine structure splitting $O(m_e\alpha^4)$
but larger than, $O(m_e \frac{m_e}{m_p}\alpha^4)$ hyperfine splitting.

☆ 

is evaluated similarly.

Now $eQ_e \frac{\vec{p}_e}{m_e}$ is replaced $(eQ_p)\frac{\vec{p}_p}{m_p}$.

$\vec{p}_e \sim -\vec{p}_e$ but $m_e \ll m_p$.

$\Longrightarrow$ negligible.

☆ selection rule in the dipole transition

$L_f \otimes \underset{\text{dipole}}{(L=1)} \otimes L_i$ irreducible decomposition of $SU(2) \sim SO(3)$ rotation.

should contain $(L=0)$ component.

$$\Rightarrow L_i \otimes 1 = \begin{cases} (L_i+1) \oplus (L_i) \oplus (L_i - 1) & \text{if } L_i \geq 1 \text{ (p-wave or higher)} \\ 1 & \text{if } L_i = 0 \text{ (s-wave)} \end{cases}$$

$$\Rightarrow \boxed{\overset{\text{any}}{}|L_f - L_i| \leq 1 \quad \text{except } L_i = L_f = 0}$$

§ 5 Unitarity

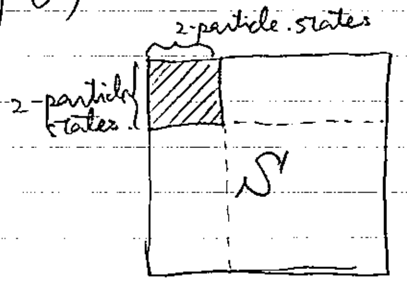
constraints: $\begin{cases} \cdot \text{ the } S\text{-matrix should be unitary} \\ \cdot\ \mathcal{H}_{\text{phys}} \text{ should have a positive definite norm} \end{cases}$

(no negative norm states)

§ 5.1 Partial wave unitarity

☆ In a finite-dim unitary matrix $U = \begin{bmatrix} u_{ij} \end{bmatrix}$ (so $U^\dagger U = \mathbb{1}$)

$$\begin{cases} \cdot\ \sum_i^{\text{all}} |u_{ij}|^2 = 1 \\ \cdot\ \sum_{\text{sum } i\text{'s}} |u_{ij}|^2 \le 1 \end{cases}$$ for any j (column vector of U)



☆ Apply this observation to the S-matrix

- $\sum_{\beta \text{ all}} |S_{\beta\alpha}|^2 = 1$.
- $\sum_{\beta:\text{2-particle state}} |S_{\beta\alpha}|^2 \le 1$.

α : some 2-particle state

"sum" = integral over 2-particle states can be made discrete (= sum).

$$\boxed{S^{2\to2}_{\beta\alpha}} =: (2\pi)^4\delta^4(p_\beta - p_\alpha) \times \left(S^{2\to2}_{\beta\alpha\,\mathrm{red}}\right) \times \sqrt{\frac{v_{34}}{v_3 v_4}} \times \sqrt{\frac{v_{12}}{v_1 v_2}}\, (4\pi)^2 .$$

This is to make sure that (when α is in the CM frame)

$$\sum_\beta \left(S^{2\to2}_{\beta\gamma}\right)^*\left(S^{2\to2}_{\beta\alpha}\right) = (2\pi)^4\delta^4(p_\gamma - p_\alpha)\sqrt{\frac{v_{56}}{v_5 v_6}}(4\pi)\int\frac{d^3\vec{p}_3}{(2\pi)^3}\int\frac{d^3\vec{p}_4}{(2\pi)^3}\frac{1}{(2E_3)(2E_4)} \times \left(\frac{v_{34}}{v_3 v_4}(4\pi)^2\right)$$
$$\times \left(S^{2\to2}_{\beta\gamma\,\mathrm{red}}\right)^*\left(S^{2\to2}_{\beta\alpha\,\mathrm{red}}\right) \times \sqrt{\frac{v_{12}}{v_1 v_2}}(4\pi) \times (2\pi)^4\delta^4(p_\beta - p_\alpha)$$

$$= (2\pi)^4\delta^4(p_\gamma - p_\alpha)\sqrt{\frac{v_{56}}{v_5 v_6}}\sqrt{\frac{v_{12}}{v_1 v_2}}(4\pi)^2\int d^2\Omega(\hat{p}) \left(S^{2\to2}_{\beta\gamma\,\mathrm{red}}\right)^*\left(S_{\beta\alpha\,\mathrm{red}}\right).$$

$\left(S^{2\to2}_{\beta\alpha\,\mathrm{red}}\right)$ in the CM frame : fn of $\underbrace{(\hat{p}_3), (\hat{p}_1)}_{\text{direction}}$ and the CM energy $(E_1 + E_2)$.

So, when α, β are 2-particle states of a pair of spin=0 fields,

$$\left(\frac{S_{\beta\alpha}^{2\to 2}}{\text{red}}\right)_{CM} = \sum_{l,m} Y_{l,m}(\hat{p}_3)\, e^{2i\delta_l(E)}\, Y_{l,m}(\hat{p}_1)^*$$

$$\left(\begin{array}{l} \cdot \int d^2\Omega\, Y_{l,m}^*(\theta,\phi)\, Y_{l',m'}(\theta,\phi) = \delta_{ll'}\,\delta_{m,m'} \\ \cdot\ Y_{l,0}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}}\, P_l(\cos\theta) \qquad \cdot\ P_l(\cos\theta=+1) = +1\ (\forall l) \end{array}\right)$$

☆ block diagonal in l due to the conservation of angul. mom.

☆ $e^{2i\delta_l(E)}$ is indep. of m : Schur's lemma

☆ S-matrix unitarity $\Longrightarrow$ $\left|e^{2i\delta_l(E)}\right| \leq 1$

$$\boxed{\text{If } \sum_{\substack{\text{2-particle} \\ \text{pairs} \\ \text{from } \alpha}} \left(\left| e^{2i\delta_l(E)} \right|\right) < 1 \quad \text{then} \quad \sum_{\gamma} \left| S_{\gamma\alpha}^{2\to\text{many}} \right|^2 > 0 .}$$

$\delta_l(E_{CM})$: "phase" shift (not a pure phase in QFT any more)

☆ 2-particle states with arbitrary spin
$\Longrightarrow$ conserved $\vec{J}$ instead of $\vec{L}$ (see Weinberg I. §3.7)

$$d\sigma^{2\to 2}_{@CM} = \frac{1}{2E_1\, 2E_2\, v_{12}} \int \frac{d^3\vec{p}_3}{(2\pi)^3} \frac{d^3\vec{p}_4}{(2\pi)^3} \frac{(2\pi)^4 \delta^4(p_3+p_4-p_1-p_2)}{2E_3\, 2E_4} |\mathcal{M}|^2$$

$$= \frac{1}{2E_1\, 2E_2\, v_{12}} \int \frac{d\vec{p}_3}{(2\pi)^3} \frac{(2\pi)\,\delta(E_3+E_4-E_1-E_2)}{2E_3\, 2E_4} \left[(4\pi)^2 \frac{v_{34}}{v_3 v_4}\right]\left[(4\pi)^2 \frac{v_{12}}{v_1 v_2}\right] \left|\mathcal{M}_{red}\right|^2$$

$$= \frac{4\pi^2}{p_1^2} \int d^2\Omega_{(\hat{p}_3)} \left| i\mathcal{M}_{red} = \left(\sum_l Y_{l,m}(\theta,\phi)\left(e^{2i\delta_l}-1\right) Y_{l,m}(\hat{p}_1)\right)\right|^2$$

(see above) $\left(\begin{array}{l} \hat{p}_1 = \hat{e}_z \\ \Rightarrow Y_{l,m} = \delta_{m,0} \times \sqrt{\frac{2l+1}{4\pi}} \end{array}\right)$

$$= \frac{\pi}{p_1^2} \sum_{l=0}^{\infty} (2l+1) \left[2\sin(\delta_l(E_{CM}))\right]^2 = \boxed{\frac{4\pi}{p_1^2} \sum_{l=0} (2l+1)\, \sin^2(\delta_l(E))}$$

(as in QM)

$$\underline{\underline{S}}^{2\to 2}_{\beta\alpha\,\text{red}} = \mathbb{1} + i\,\underline{\underline{\mathcal{M}}}^{2\to 2}_{\beta\alpha\,\text{red}}$$

$\hookrightarrow$ defined similarly to $\underline{\underline{S}}^{2\to 2}_{\text{red}}$

* In the weak coupling regime, tree level computation $\Rightarrow$ $i\mathcal{M}$: small dimensionless Im part (small phase shift in $\underline{\underline{S}}^{2\to 2}_{\text{red}}$)

most of the examples in the class so far

* When one contribution to $\underline{\mathcal{M}}^{2\to 2}_{\beta\alpha}$ is too much,

$$\left| 1 + \frac{R}{32\pi^2}\sqrt{\frac{4\pi}{2l+1}} \int d\Omega\, \Delta(i\mathcal{M}(\theta,\phi))\left(Y^*_{l,m}(\theta\phi)\right) \right| > 1 \quad \text{for any one of } l$$

then there must be other important contributions to $\mathcal{M}^{2\to 2}_{\beta\alpha}$ that restore unitarity.

example: $\mathcal{L} = G_F(\bar{\psi}\psi)(\psi\psi)$ — (4-fermi theory)

$\mathcal{L} = \frac{1}{M}(lh)(lh)$ (Majorana neutrino mass)

$\mathcal{M} \sim (G_F \cdot E_{CM}^2)$

$\mathcal{M} \sim \left(\frac{E_{CM}}{M}\right)$

something has to happen @ $\begin{cases} E \sim 1/\sqrt{G_F} \\ E \sim M \end{cases}$