§5.2 Optical theorem

The unitarity of the S-matrix also implies

$$\mathbb{1} = S^\dagger S = \left(\mathbb{1} + (2\pi)^4 \delta^4(p_\gamma - p_\beta)\left(-i\mathcal{M}^*_{\beta\gamma}\right)\right)\left(\mathbb{1} + (2\pi)^4\delta^4(p_\beta - p_\alpha)\, i\mathcal{M}_{\beta\alpha}\right)$$

$$= \mathbb{1} + (2\pi)^4\delta^4(p_\gamma - p_\alpha)\left\{ i\left[\mathcal{M}_{\gamma\alpha} - (\mathcal{M}_{\alpha\gamma})^*\right] + \sum_\beta (\mathcal{M}_{\beta\gamma})^*(\mathcal{M}_{\beta\alpha})(2\pi)^4\delta^4(p_\beta - p_\alpha)\right\}$$

so

$$\boxed{\frac{\mathcal{M}_{\gamma\alpha} - (\mathcal{M}_{\alpha\gamma})^*}{(i)} = \int \prod_{j=1}^{N_\beta}\left[\frac{d^3\vec{p}_j}{(2\pi)^3}\frac{1}{(2E_j)}\right](2\pi)^4\delta^4(p_\beta - p_\alpha)\,(\mathcal{M}_{\beta\gamma})^*(\mathcal{M}_{\beta\alpha})}$$

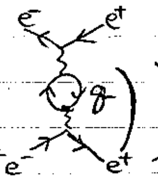
As a particular case $\gamma = \alpha$,

$$\boxed{2\,\mathrm{Im}(\mathcal{M}_{\alpha\alpha}) = \int \prod_{j=1}^{N_\beta}\left[\frac{d^3\vec{p}_j}{(2\pi)^3}\frac{1}{(2E_j)}\right](2\pi)^4\delta^4(p_\beta - p_\alpha)\,\left|\mathcal{M}_{\beta\alpha}\right|^2 = \begin{cases}\sigma_{tot}\cdot(4E_1E_2v_{12}) \\ \text{or} \\ \Gamma_{tot}\cdot(2E)\end{cases}}$$

optical theorem

Useful because....

☆ $\boxed{\mathrm{Im}(\mathcal{M}_{\alpha\alpha}) \Longrightarrow \Gamma_{tot},\ \sigma_{tot}}$

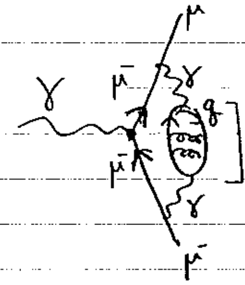
$e^+ + e^- \to \gamma$ or $Z \to$ hadrons $\left(\sigma_{tot}\cdot s\right) \cong 2\,\mathrm{Im}\Big[\mathcal{M}\Big($  $\Big)\Big]$

at $E_{CM} \gg$ GeV

[Perturbative calculations are available for "inclusive enough" observables.]
such as σ_{tot}.

☆ $\boxed{\Gamma_{tot},\ \sigma_{tot} \Longrightarrow \mathrm{Im}(\mathcal{M}_{\alpha\alpha})}$

to estimate contributions to anomalous magnetic moment of μ^-,



we can use $[\sigma_{tot}(e^+ + e^- \to \text{hadron})\cdot s]$ to determine $\mathrm{Im}(\mathcal{M}_{\gamma^*\gamma^*})$.
↖ unitarity

and $\mathrm{Re}(\mathcal{M}_{\gamma^*\gamma^*})(s + i\varepsilon) = \int \frac{ds'}{\pi}\frac{\mathrm{Im}(\mathcal{M}_{\gamma^*\gamma^*}(s'))}{s' - s}$.

dispersion integral ($\mathcal{M}(s + i\varepsilon)$: holomorphic in s)

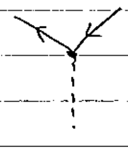
[Kramers–Kronig relation]

☆ $\boxed{\sigma_{tot} \longrightarrow \mathrm{Im}(\mathcal{M}_{\alpha\alpha})}$ for perturbative calculation.

Think of a theory with $\mathcal{L} = \mathcal{L}_{kin} + g\,\phi\,\bar{\Psi}\Psi$. ϕ: scalar

$$\Rightarrow \int \frac{d^3\vec{p}_\Psi}{(2\pi)^3}\frac{1}{2E_{p_\Psi}} \int \frac{d^3\vec{p}_{\bar{\Psi}}}{(2\pi)^3}\frac{1}{2E_{p_{\bar{\Psi}}}} (2\pi)^4\delta^4(p_\Psi + p_{\bar{\Psi}} - p_\phi) \underset{\text{spin-sum}}{\sum} |\mathcal{M}|^2 = \frac{g^2}{4\pi}\left\{E^2 - (2m_\Psi)^2\right\}.$$

$$\left[\begin{array}{l} i\mathcal{M}(\phi \longrightarrow \Psi + \bar{\Psi}) \\ \quad\Downarrow \\ \quad (E, \vec{0})\ \text{CM frame} \end{array}\right. = $$



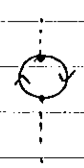
straightforward calculation. ⟨tree-level⟩.

So...

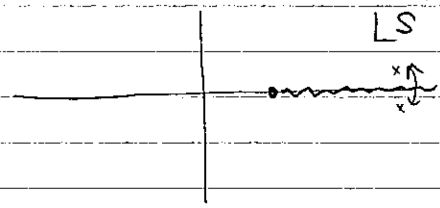
$$\overbrace{\underline{2\,\mathrm{Im}\left[\mathcal{M}(\phi \to \phi)\right] \Leftarrow \frac{g^2}{4\pi}\left\{s - (2m_\Psi)^2\right\}}}\quad \text{unitarity}$$

$$\mathcal{M}(\phi \to \phi)\ \text{at}\ s = \frac{-g^2}{8\pi^2}\left\{s - (2m_\Psi)^2\right\}\ln\left(\frac{(2m_\Psi)^2 - s - i\varepsilon}{(2m_\Psi)^2}\right) + \left(\begin{array}{l}\text{rational real} \\ (\text{for real } s)\end{array}\right)$$

$\|$



$\cdot\,(i)$

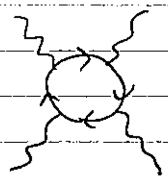


$\left[\begin{array}{l}\text{branch cut} \\ \text{at } (2m_\Psi)^2 \le s\end{array}\right]$

We have managed to obtain an expression for a 1-loop graph

$\boxed{\text{without doing 1-loop computation.}}$

More generally...



$= i\mathcal{M}(s, t)$ should be a holomorphic function of (s, t) except poles and branch cuts

that satisfy all of $\left\{\begin{array}{l} s\text{-channel} \\ t\text{-channel} \\ u\text{-channel}\end{array}\right\}$ unitarity relation.

§6 Low-energy effective theory

Here, "theory" is in the sense of <u>model</u>.

$\boxed{\text{Example 1}}$ QED with $\gamma, e^{\pm}, \mu^{\pm}$ $\longrightarrow$ QED with $\gamma, e^{\pm}$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}_{(e)}(i\gamma^{\mu}D_{\mu} - m_{(e)})\Psi_{(e)} + \bar{\Psi}_{(\mu)}(i\gamma^{\mu}D_{\mu} - m_{(\mu)})\Psi_{(\mu)} \qquad (*)$$

Start from a theory (=model) above.

If we are interested in physics with energy below $m_{(\mu)} \sim 106$ MeV,

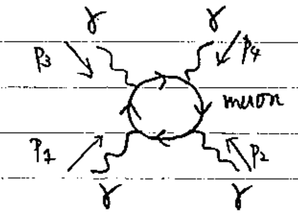
we do not have to maintain $\Psi_{(\mu)}$ in the Lagrangian.

But we have to use

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}_{(e)}(i\gamma^{\mu}D_{\mu} - m_{(e)})\Psi_{(e)} + \frac{e^4 C^{(1)}_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}}{16\pi^2 m_{(\mu)}^4} F^{\mu_1\nu_1}F^{\mu_2\nu_2}F^{\mu_3\nu_3}F^{\mu_4\nu_4}$$

$$+ \cdots \qquad (**)$$

in order to account for the $\gamma+\gamma \to \gamma+\gamma$ scattering amplitude



$$= i\mathcal{M}(m_{(\mu)}, p_i^{\mu}, \varepsilon_i^{\nu})$$

expand in power series of $p/m_{(\mu)}$

$$= i\frac{e^4 C^{(1)}_{\mu_1\nu_1\cdots\mu_4\nu_4}}{16\pi^2 m_{(\mu)}^4}(p_1^{\mu_1}\varepsilon_1^{\nu_1} - p_1^{\nu_1}\varepsilon_1^{\mu_1})(p_2^{\mu_2}\varepsilon_2^{\nu_2} - p_2^{\nu_2}\varepsilon_2^{\mu_2})(p_3^{\mu_3}\varepsilon_3^{\nu_3} - p_3^{\nu_3}\varepsilon_3^{\mu_3})(p_4^{\mu_4}\varepsilon_4^{\nu_4} - p_4^{\nu_4}\varepsilon_4^{\mu_4})$$

$$+ \mathcal{O}\left(p^6/m_{(\mu)}^6\right)$$

<u>The latter (**) is the low-energy effective theory (=model) of the former (*).</u>

The latter theory with just the $\mathcal{O}(1/m_{(\mu)}^4)$ term

will violate partial wave unitarity at $E \sim m_{(\mu)}$.

But all the terms in the $p/m_{(\mu)}$ expansion are equally important

in the partial wave unitarity at $E \sim m_{(\mu)}$.

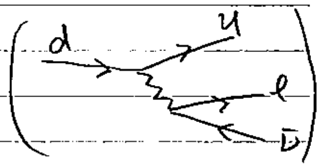
We should use the high-energy theory (*) at $E \gtrsim m_{(\mu)}$

$\boxed{\text{Exampl 2}}$ The Standard Model $\longrightarrow$ QCD + QED + 4-fermi term

$$\mathcal{L}_i \supset \left[\bar{u}_i\, i\gamma^\mu\left(\frac{1-\gamma_5}{2}\right) i g_w \left(\frac{A^1_\mu + iA^2_\mu}{2}\right) d_j\right] V_{ij} - \frac{1}{4g_w^2}\left(\underbrace{F^1_{\mu\nu}F^{1\mu\nu} + F^2_{\mu\nu}F^{2\mu\nu}}_{\text{W-boson kinetic term}} + F^3_{\mu\nu}F^{3\mu\nu}\right)$$
$$+ \left[\bar{l}\, i\gamma^\nu \left(\frac{1-\gamma_5}{2}\right) i g_w \left(\frac{A^1_\nu - iA^2_\nu}{2}\right)\nu\right] + \cdots \qquad \text{———}(*)$$

- i: subscripts in u_i, d_i $\Rightarrow$ generation $(i = 1, 2, 3)$
- V_{ij} : a 3×3 unitary matrix (called Cabbibo Kobayashi Maskawa (CKM) matrix)
- In the Peskin-Schroeder convention,

$$\gamma^0 = \begin{pmatrix} & \mathbb{1} \\ \mathbb{1} & \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \tau^i \\ -\tau^i & 0 \end{pmatrix} \quad \gamma_5 := \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$



This high-energy theory yields

$$i\mathcal{M}(d_j \to u_i + l + \bar\nu) = \left[\bar{u}(\vec{p}_{u_i})\gamma^\mu(1-\gamma_5)u(\vec{p}_{d_j})\right] V_{ij} \times \frac{\left(\frac{ig_w}{4}\right)\left(\frac{-ig_w}{4}\right)\left(-i\times\eta_{\mu\nu}\times 2\right)}{(p_d-p_u)^2 - m_W^2 + i\varepsilon}$$
$$\left[\bar{u}(\vec{p}_e)\gamma^\nu(1-\gamma_5)v(\vec{p}_\nu)\right].$$

Since $m_W \simeq 80$ GeV, it makes sense (when applied to low-energy physics) to expand in P/m_W

$$\frac{1}{(p_d-p_u)^2 - m_W^2} \Rightarrow \frac{1}{-m_W^2} - \frac{(p_d-p_u)^2}{(m_W^2)^2} - \frac{(p_d-p_u)^4}{(m_W)^6} - \cdots$$

and retain just a few terms.

The LO term in the amplitude is reproduced by

$$\mathcal{L}_{int} \mathrel{+}\cong -\left(\frac{g_w^2}{8m_W^2}\right)\left[\bar{u}_i\gamma^\mu(1-\gamma_5)d_j\right]\left[\bar{l}\gamma_\mu(1-\gamma_5)\nu\right]V_{ij} \qquad \text{———}(**)$$

to $\mathcal{L}_{QED\times QCD}$ (without $W^\pm$ bosons)

called 4-fermi interaction

$\boxed{\text{(**) should be modified @ 1-loop.}}$

Example 3 = Question (homework) The seesaw mechanism simplified

In a theory with one scalar ϕ and two Dirac fermions Ψ and Ψ',

suppose that $\mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi) + \bar{\Psi} i \gamma^\mu (\partial_\mu \Psi) + \bar{\Psi}'(i\gamma^\mu \partial_\mu - M)\Psi'$

$$+ \lambda \phi \bar{\Psi} \Psi' + \lambda^* \phi \bar{\Psi}' \Psi . \quad \text{——} (*)$$

So, ϕ and Ψ are massless, but Ψ' is massive.

Now, verify that the low-energy effective theory of $(*)$ at $E \ll M$ is given by

$$\mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi) + \bar{\Psi} i \gamma^\mu (\partial_\mu \Psi) + \frac{|\lambda|^2}{M} \phi \bar{\Psi} \Psi \phi . \quad \text{——} (**)$$

analogy: ϕ: Higgs doublet

Ψ: left-handed neutrino

Ψ': right-handed neutrino

$$\Big(\underbrace{\text{The SM} + \text{RH}\nu + \nu\,\text{Yukawa int.}}_{(*)} \Big) \longrightarrow \Big(\underbrace{\text{The SM} + \frac{|\lambda|^2}{M}(lh)(lh)}_{(**)} \Big)$$

In this case (the seesaw mechanism), we should deal with

- ϕ as a complex boson
- Ψ, Ψ' as Weyl spinors (2-component spinors)

in fact.

- low-energy approximation
- derivative/mass expansion (and truncation)
- Born – Oppenheimer approximation.

Example 4 QCD $\longrightarrow$ hadrons

quark + gluon $\rightarrow$ $\mathcal{L} = (\partial_\mu \pi^a)(\partial^\mu \pi^a) + \cdots$ — (**)

$$- (\partial_\mu \rho_\nu^a - \partial_\nu \rho_\mu^a)(\partial^\mu \rho^{a\nu} - \partial^\nu \rho^{a\mu}) - \frac{1}{2} m^2 \rho_\mu^a \rho^{a\mu}$$

$$+ \cdots\cdots$$

(perturbative calculation cannot determine all the information of (**).)

Example 5 quantum hall system (e^-'s in $\langle \vec{B} \rangle \neq 0$ in 2+1 dim) $\longrightarrow$ Chern-Simons theory ??

[ask condensed matter physicists.]

Example 6 QED $(\gamma, e^\pm)$ $\longrightarrow$ $E \ll m_e$

pair creation cannot take place anymore.

- Initial states with just photons $\Rightarrow$ an effective theory of γ.
- Initial states with just one e^- (+ γ's ?)

$$\Rightarrow \mathcal{L} \cong \psi^\dagger \left(i\partial_t - m - eQ_e\varphi - \frac{(i\vec{\partial} + eQ_e\vec{A})^2}{2m_e} - \cdots \right) \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$ — (**)

This two component "ψ" is a part of the 4-component as

$$\underset{\text{4 componet}}{\Psi(\vec{p}, t)} \cong \begin{pmatrix} \psi(\vec{p}, t) \\ \frac{\vec{p}\cdot\vec{\sigma}}{2m_e} \psi(\vec{p}, t) \end{pmatrix} + (\text{positron}).$$

- Even furtherer in low-energy (so there is no ionization)

$\Rightarrow$ an effective theory of bound states — (***)

$$\mathcal{L} = \phi_{1s}^\dagger \left(i\partial_t - \frac{(i\vec{\partial})^2}{2m_{1s}} + \cdots \right) \phi_{1s} + \vec{\phi}_{2p}^\dagger \left(i\partial_t - \frac{(i\vec{\partial})^2}{2m_{2p}} + \cdots \right) \vec{\phi}_{2p} + \cdots$$

$$+ \mathcal{L}_{int.} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Sometimes the low-energy effective theory can be just an empty system.

Supplementary notes

Dirac spinor field: a 4-component field $\Psi(x)$ that transforms under Lorentz transformation $\Lambda^{\alpha}{}_{\beta} = \exp\left[\frac{\omega_{\mu\nu}}{2}\left[\eta^{\mu\alpha}\delta^{\nu}{}_{\beta} - \eta^{\nu\alpha}\delta^{\mu}{}_{\beta}\right]\right]$

$$\Psi(x) \longrightarrow \Psi'(x) = \exp\left[\frac{\omega_{\mu\nu}}{8}[\gamma^{\mu},\gamma^{\nu}]\right]\Psi(\Lambda^{-1}\cdot x).$$ ($\omega_{\mu\nu}$: anti-symmetric 4×4.)

This $\exp\left[\frac{\omega_{\mu\nu}}{8}[\gamma^{\mu},\gamma^{\nu}]\right]$ is a 4-dimensional representation of the Lorentz group $SO(3.1)$, but it is not irreducible.

When we choose $\gamma^{\mu} = \begin{pmatrix} 0 & \sigma^{\mu} \\ \bar{\sigma}^{\mu} & 0 \end{pmatrix}_{4\times4}$ with $\sigma^{\mu} = (\mathbb{1}, \vec{\tau})$, $\bar{\sigma}^{\mu} = (\mathbb{1}, -\vec{\tau})$ —— (**)

$[\gamma^{\mu},\gamma^{\nu}]$ is block diagonal for any pair of (μ,ν) $(\mu\neq\nu)$

$\Psi = \begin{pmatrix}\psi_{\alpha} \\ \bar{\chi}^{\dot{\alpha}}\end{pmatrix}$

$$\psi_{\alpha}(x) \longrightarrow \psi'_{\alpha}(x) = \exp\left[\frac{\omega_{\mu\nu}}{8}(\sigma^{\mu}\bar{\sigma}^{\nu} - \bar{\sigma}^{\mu}\sigma^{\nu})\right]_{\alpha}{}^{\beta}\psi_{\beta}(\Lambda^{-1}\cdot x) \qquad \text{—— (*L)}$$

$$\bar{\chi}^{\dot{\alpha}}(x) \longrightarrow \bar{\chi}^{\dot{\alpha}\prime}(x) = \exp\left[\frac{\omega_{\mu\nu}}{8}(\bar{\sigma}^{\mu}\sigma^{\nu} - \bar{\sigma}^{\nu}\sigma^{\mu})\right]^{\dot{\alpha}}{}_{\dot{\beta}}\bar{\chi}^{\dot{\beta}}(\Lambda^{-1}\cdot x) \qquad \text{—— (*R)}$$

Weyl spinor field is a 2-component field that transforms as either one of (*L) or (*R). ~~The transformation laws (*L) and (*R)~~ are not the same in 3+1 dimensions.

The Dirac Lagragian

$$\mathcal{L} = \bar{\Psi}(i\gamma^{\mu}\partial_{\mu} - m)\Psi = \bar{\psi}_{\dot{\alpha}}(i\bar{\sigma}^{\mu})\partial_{\mu}\psi_{\alpha}) + \chi^{\alpha}(i\sigma^{\mu})\partial_{\mu}\bar{\chi}^{\dot{\alpha}}) - m(\chi^{\alpha}\psi_{\alpha} + \bar{\psi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}})$$

$\left(\Psi = \begin{pmatrix}\psi_{\alpha} \\ \bar{\chi}^{\dot{\alpha}}\end{pmatrix} \quad \bar{\Psi} = (\chi_{\alpha}, \bar{\psi}^{\dot{\alpha}}) \quad \bar{\psi}^{\dot{\alpha}} = \varepsilon^{\dot{\alpha}\dot{\beta}}(\psi_{\beta})^{\dagger}, \ \chi_{\alpha} = \varepsilon_{\alpha\beta}(\bar{\chi}^{\dot{\beta}})^{\dagger}\right)$ $\varepsilon_{\alpha\beta}$: antisymmetric 2×2 matrix

In a theory (model) where $\psi_{\alpha} = \chi_{\alpha}$. ($\Leftarrow$ this is not the Dirac theory anymore. it is another theory.)

$$\boxed{\mathcal{L} = \bar{\psi}_{\dot{\alpha}}\, i(\bar{\sigma}^{\mu})^{\dot{\alpha}\alpha}(\partial_{\mu}\psi) - \frac{1}{2}m(\psi^{\alpha}\psi_{\alpha} + \bar{\psi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}})} \longleftarrow \text{This is for neutrinos.}$$

~~The $\psi_{\alpha} = \chi_{\alpha}$ condition is equivalent to the Majorana condition.~~

$\boxed{\Psi^{cc} = e^{i\alpha}(\gamma^{\mu=2})\Psi}$ for some complex phase $e^{i\alpha}$.

We can impose this condition because $(\gamma^{\mu=2})^{-1}(\gamma^{\nu})^{cc}(\gamma^{\mu=2}) = -\gamma^{\nu}$ (use (**) to verify)

and hence $\exp\left[\frac{\omega_{\mu\nu}}{8}[\gamma^{\mu},\gamma^{\nu}]\right]^{cc}\gamma^{2} = \gamma^{2}\exp\left[\frac{\omega_{\mu\nu}}{8}[\gamma^{\mu},\gamma^{\nu}]\right]$.