Setting the initial state to the ground state (vacuum) or to the thermal distribution.

- $\langle q_f \text{ at } t_f \mid q_i \text{ at } t_i \rangle = \sum_n \psi_n(q_f)\,\psi_n^*(q_i)\, e^{-iE_n(t_f-t_i)}$.

$$[t_i \to -\infty] \Longrightarrow t_i \to -\infty(1-i\varepsilon) \qquad (i\varepsilon \text{ prescription})$$
$$[t_f \to +\infty] \Longrightarrow t_f \to +\infty(1-i\varepsilon).$$

$$\times\ e^{-i[(\Delta E)>0][(-i\varepsilon)\infty]} = e^{-\infty\cdot\varepsilon\cdot(\Delta E)>0}$$

excited state contributions die out.

As a result.

$$\boxed{\begin{aligned}&\int_{i\varepsilon\text{-prescription}} \mathcal{D}q\,\mathcal{D}p \quad e^{i\int dt(p\dot{q}-H)} \quad \prod_i f_i(q_i) \ \prod_j \tilde{f}_j(p_j) \\ &\propto \langle 0.\text{future} \mid T\{\prod_i f_i(q(t_i))\ \prod_j \tilde{f}_j(p(t_j))\} \mid 0.\text{past}\rangle\end{aligned}}$$

operators in the Heisenberg picture.

- Boltzmann weight.

$$Z = \sum_n e^{-\beta E_n} = \int dq \ \sum_n \psi_n(q)\,\psi_n^*(q)\ e^{-\beta E_n}$$

$$= \int d(q_i = q_f)\ \langle q_f \text{ at } (t_i - i\beta) \mid q_i \text{ at } t_i \rangle$$

$$= \int_{q(t=-i\beta)=q(t=0)} \mathcal{D}q(\tau)\,\mathcal{D}p(\tau)\ e^{i\sum_{k=1}^{N}\{p_k(q_k-q_{k-1}) - (\Delta t = -i\Delta\tau)H(p_k,q_{k-1})\}}$$

$$= \int_{q(\tau=\beta)=q(\tau=0)} \mathcal{D}q(\tau)\mathcal{D}p(\tau)\ e^{\int d\tau(ipq' - H)}$$

$$\left[\frac{\partial}{\partial\tau} \Rightarrow {}' \quad t = -i\tau\right]$$

If $H = \frac{p^2}{2m} + V(q)$ then

$$Z \propto \int_{q(\tau=\beta)=q(\tau=0)} dq \quad e^{-\int d\tau \left\{ \frac{m}{2} (q')^2 + V(q) \right\}}$$

after Gaussian integral.

§6.2 Path integral formulation for systems with a fermion

Consider a quantum mechanical system with only 2 states.

$$\{b, b^\dagger\} = 1 \qquad |1\rangle = b^\dagger |0\rangle$$
$$|0\rangle = b|1\rangle$$

States $\Leftrightarrow$ $c_0|0\rangle + c_1|1\rangle$ $\quad (c_0, c_1) \in \mathbb{C}^2$

Let us introduce a notion "Grassmann variables".

Instead of $q_i q_j = q_j q_i$, $\quad \theta_i \theta_j = -\theta_j \theta_i$.

In particular, $\theta_i\theta_i = -\theta_i\theta_i = 0$.

The Taylor series expansion of $f(q_i\text{'s}, \theta_i\text{'s})$:

$$f = f_0(q\text{'s}) + \sum_i f_i(q\text{'s})\,\theta_i + \sum_{i<j} f_{ij}(q)\,\theta_i\theta_j + \sum_{i<j<k} f_{ijk}(q)\,\theta_i\theta_j\theta_k$$
$$+ \cdots \qquad + f_{\text{all}}(q)\cdot\Big(\prod_i \theta_i\Big)$$

Define integrals over Grassmann variables by

$$\boxed{\int d\theta\, f(q,\theta) = \int d\theta\,\big(f_0(q) + \theta f_1(q)\big) := f_1(q).}$$

$$\int d\theta_2\, d\theta_1\, f(q, \theta_1, \theta_2) = \int d\theta_2 \int d\theta_1 \big(f_0 + f_1\theta_1 + f_2\theta_2 + f_{12}\theta_1\theta_2\big)$$
$$= \int d\theta_2\,(f_1 + f_{12}\theta_2) = f_{12}(q)$$

note that $d\theta_2\, d\theta_1 = -d\theta_1\, d\theta_2$.

The "wave function" of ~~the~~ a state $|\text{state}\rangle = c_0|0\rangle + c_1|1\rangle$

can be expressed as $\Psi(\bar{\theta}) = c_0 + \bar{\theta} c_1$.

by using a Grassmann variable $\bar{\theta}$.

As an analogy to the relation $\Psi(q) = \langle q|\text{state}\rangle$.

of a bosonic system, we can ~~think~~ use

$$\Psi(\bar{\theta}) = \left(\langle 0| + \bar{\theta}\langle 1|\right)|\text{state}\rangle.$$

↳ (formally regarded as a state where the linear combination coefficients can be Grassmann ....)

A wave function $\Psi(q)$ can be cast back into a state

through $|\text{state}\rangle = \int dq\ |q\rangle \Psi(q)$.

Its analogy is

$$|\text{state}\rangle = \int d\bar{\theta}\, d\theta \left(|0\rangle + |1\rangle \theta\right) e^{-\bar{\theta}\theta}\, \Psi(\bar{\theta})$$

Time evolution of wave functions in this 2-state system can be described in terms of path integral as follows.

$$\Psi(\bar{\theta}_N @ t_N) = \left(\langle 0| + \bar{\theta}_N \langle 1|\right) e^{-iHT} \underset{@\, t_0}{|\text{state}\rangle}$$

$$= \left(\langle 0| + \bar{\theta}_N \langle 1|\right) e^{-iH(\Delta t)} \int d\bar{\theta}_{N-1} d\theta_{N-1} \left(|0\rangle + |1\rangle \theta_{N-1}\right) e^{-\bar{\theta}_{N-1}\theta_{N-1}} \left(\langle 0| + \bar{\theta}_{N-1}\langle 1|\right)$$

$$e^{-iH(\Delta t)} \int d\bar{\theta}_{N-2} d\theta_{N-2} \left(|0\rangle + |1\rangle \theta_{N-2}\right) e^{-\bar{\theta}_{N-2}\theta_{N-2}} \left(\langle 0| + \bar{\theta}_{N-2}\langle 1|\right)$$

$$\vdots$$

$$e^{-iH(\Delta t)} \int d\bar{\theta}_1 d\theta_1 \left(|0\rangle + |1\rangle \theta_1\right) e^{-\bar{\theta}_1\theta_1} \left(\langle 0| + \bar{\theta}_1\langle 1|\right)$$

$$e^{-iH\Delta t} |\text{state}\rangle.$$

Let $H = H_{00} + H_{01} b + \cancel{H_{10}b^\dagger}\ b^\dagger H_{10} + H_{11} b^\dagger b.$ (b and $b^\dagger$ are operators)

Then, modulo ~~up to~~ $\mathcal{O}((\Delta t)^2)$ terms

$$\left(\langle 0| + \bar{\theta}_k \langle 1|\right) e^{-iH(\Delta t)} \left(|0\rangle + |1\rangle \theta_{k-1}\right)$$

$$\cong \left(1 + \bar{\theta}_k \theta_{k-1}\right) - i(\Delta t)\left(H_{00} + H_{01}\theta_{k-1} + \bar{\theta}_k H_{10} + \bar{\theta}_k H_{11}\theta_{k-1}\right)$$

$$\cong \exp\left[\bar{\theta}_k \theta_{k-1} - i(\Delta t)\underbrace{\left(H_{00} + H_{01}\theta_{k-1} + \bar{\theta}_k H_{10} + \bar{\theta}_k \theta_{k-1} H_{11}\right)}_{\parallel}\right].$$

H w. $b^\dagger$ and b replaced by Grassmann variables $\bar{\theta}_k$ and θ_{k-1}.

So,

$$\Psi(\bar{\theta}_N @ t_N) = \int d\bar{\theta}_{N-1} d\theta_{N-1} \cdots d\bar{\theta}_1 d\theta_1 \exp\left[\sum_{k=0}^{N-1}\left(\bar{\theta}_{k+1}\theta_k\right) - \sum_{k=0}^{N-1}\left(\bar{\theta}_k\theta_k\right) - i\Delta t \sum_{k=0}^{N-1} H\begin{pmatrix} b^\dagger \to \bar{\theta}_{k+1} \\ b \to \theta_k \end{pmatrix}\right]$$

$$d\bar{\theta}_0 d\theta_0 \quad \Psi_{in}(\bar{\theta}_0 @ t_0).$$

$$\simeq \int d\bar{\theta}_{N-1} \sim d\theta_0 \exp\left[i \int dt \left\{-i\dot{\bar{\theta}})\theta - H(b^\dagger \to \bar{\theta}, b \to \theta)\right\}\right] \Psi_{in}(\bar{\theta}_0 @ t_0)$$

$$\simeq \int d\bar{\theta}_{N-1} \sim d\theta_0 \exp\left[i \int dt \left\{\bar{\theta}(i\partial_t \theta) - H(b^\dagger \to \bar{\theta}, b \to \theta)\right\}\right] \Psi_{in}(\bar{\theta}_0 @ t_0)$$

(cf. $L = \bar{\theta}(i\partial_t - \mathcal{H})\theta.$)

Using the relation

$$\langle \text{state}' | \text{state} \rangle = c_0'^{*} c_0 + c_1'^{*} c_1$$

$$\int d\bar{\theta}\, d\theta\, \underbrace{\left(c_0'^{*} + c_1'^{*}\theta\right)}_{(\Psi_{\text{state}'})^{*}} \underbrace{\left(c_0 + \bar{\theta} c_1\right)}_{(\Psi_{\text{state}})} e^{-\bar{\theta}\theta} = \int d\bar{\theta}\, d\theta \left(c_0'^{*} c_0 (-\bar{\theta}\theta) + (c_1'^{*}\theta)(c_1\bar{\theta}) \right),$$

- $$\boxed{\begin{aligned} &\langle \text{state}'_{@t_f} |\, e^{-iH(t_f - t_i)}\, | \text{state}_{@t_i} \rangle \\ &= \int (d\bar{\theta}_N\, d\theta_N)(d\bar{\theta}_{N-1}\, d\theta_{N-1}) \cdots (d\bar{\theta}_1\, d\theta_1)(d\bar{\theta}_0\, d\theta_0)\, (\Psi^{*}_{\text{state}'})(\theta_N)\, e^{\sum^{k} \{ -\bar{\theta}_k (\Delta\theta_{k-1}) - i(\Delta t) H(\bar{\theta}_k, \theta_{k-1}) \}} \\ &\qquad (\Psi_{\text{state}})(\theta_0) \end{aligned}}$$

and the relation

$$\left(\langle 0| + \bar{\theta}_k \langle 1| \right) e^{-i(\Delta t) H}\, b \left(|0\rangle + |1\rangle \theta_{k-1} \right) = \exp\left[-i(\Delta t)\left(H_{00} + \bar{\theta}_k H_{10} \right) \right] \theta_{k-1}$$

$$= \exp\left[\bar{\theta}_k \theta_{k-1} - i\Delta t\, H\begin{pmatrix} b \to \theta_{k-1} \\ b^{\dagger} \to \bar{\theta}_k \end{pmatrix} \right] \theta_{k-1},$$

$$\left(\langle 0| + \bar{\theta}_k \langle 1| \right) b^{\dagger} e^{-i(\Delta t) H} \left(|0\rangle + |1\rangle \theta_{k-1} \right) = \bar{\theta}_k \exp\left[-i(\Delta t)\left(H_{00} + H_{01}\theta_{k-1} \right) \right]$$

$$= \bar{\theta}_k \exp\left[\bar{\theta}_k \theta_{k-1} - i(\Delta t) H\begin{pmatrix} b \to \theta_{k-1} \\ b^{\dagger} \to \bar{\theta}_k \end{pmatrix} \right],$$

- $$\boxed{\begin{aligned} &\langle \text{state}'_{@t_f} |\, T\left\{ e^{-iH(t_f - t_i)}\, b^{\dagger}(t')\, b(t'') \right\} | \text{state}_{@t_i} \rangle \\ &= \int \mathcal{D}\bar{\theta}\, \mathcal{D}\theta\, (\Psi^{*}_{\text{state}'})(\theta_N)\, e^{i\int dt \{ \bar{\theta}\, i(\partial_t \theta) - H(\bar{\theta}, \theta) \}}\, \bar{\theta}(t')\, \theta(t'')\, (\Psi_{\text{state}})(\theta_0) \end{aligned}}$$

§ 6.3 Path integral formulation for quantum field theories

Example 1: a free scalar boson on $(d+1)$-dimensional space-time

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 \iff \mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\partial_i\phi)(\partial_i\phi) + \frac{1}{2}m^2\phi^2$$

$$\left[\phi(\vec{x},t),\ \pi(\vec{y},t)\right] = i\delta^d(\vec{x}-\vec{y})$$

This is just a quantum mechanichs of infinitely many harmonic oscillators.

Individual oscillators are labeled by $\vec{k}$ $\left(\omega = \sqrt{\vec{k}^2+m^2}\right)$.

$$\boxed{\begin{aligned}&\langle\Omega|T\{\phi(x_1^\mu)\ \phi(x_2^\mu)\cdots\ \pi(y_1^\mu)\ \pi(y_2^\mu)\cdots\}|\Omega\rangle \\ &= \frac{\int_{+i\epsilon}\mathcal{D}\phi\,\mathcal{D}\pi\ e^{i\int dt\int d^dx\,(\pi\dot{\phi}-\mathcal{H}=\mathcal{L})}\left(\phi(x_1)\phi(x_2)\cdots\pi(y_1)\pi(y_2)\cdots\right)}{\int_{+i\epsilon}\mathcal{D}\phi\,\mathcal{D}\pi\ e^{i\int dt\int d^dx\,\mathcal{L}}}\end{aligned}}$$

Example 1': add interaction terms to $\mathcal{H}$ $\left(\text{eg. } += \frac{\lambda}{4!}\phi^4\right)$

$\Rightarrow$ just replace $(\pi\dot{\phi}-\mathcal{H}_0)$ by $(\pi\dot{\phi}-\mathcal{H})$

Example 2: Dirac theory $\mathcal{L} = \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi$

$$\boxed{\begin{aligned}&\langle\Omega|T\{\Psi(x_1)\Psi(x_2)\cdots\ \bar{\Psi}(y_1)\bar{\Psi}(y_2)\cdots\}|\Omega\rangle \\ &= \frac{\int_{+i\epsilon}\mathcal{D}\bar{\Psi}\mathcal{D}\Psi\ e^{i\int dt\,d^3x\,\mathcal{L}}\left(\Psi(x_1)\Psi(x_2)\cdots\bar{\Psi}(y_1)\bar{\Psi}(y_2)\cdots\right)}{\int\mathcal{D}\bar{\Psi}\mathcal{D}\Psi\ e^{i\int dt\,d^3x\,\mathcal{L}}}\end{aligned}}$$

where we integrate over infinitely many Grassmann coordinates.
$\hookrightarrow$ (labeled by $\vec{k}\in\mathbb{R}^3$, Ψ vs $\bar{\Psi}$ and time slices.)

§6.4 Partition function, linear response, functional determinant

In statistical mechanics,

$$Z := \sum_{\text{all states}} e^{-\beta E_i} = \mathrm{tr}_{\mathcal{H}}\left(e^{-\beta H}\right)$$ is called the partition function of a given system.

When $H = \int d^d\vec{x}\, \mathcal{H}(\phi, \pi)$ is given,

$$\boxed{Z(\beta) = \int_{\substack{\phi(\vec{x}, t=-i\beta) \\ =\phi(\vec{x}, t=0)}} \mathcal{D}\pi\mathcal{D}\phi\; e^{i\int_0^{-i\beta} dt \int d^d\vec{x}\,(\pi\dot{\phi} - \mathcal{H})} = \int_{\substack{\phi(\vec{x}, \tau=\beta) \\ =\phi(\vec{x}, \tau=0)}} \mathcal{D}\pi\mathcal{D}\phi\; e^{-\int_0^{\beta} d\tau \int d^d\vec{x}\left(\mathcal{H} - i\pi\partial_\tau\phi\right)} \quad \left(\left(t \to -i\tau\right)\right)}$$

trace ⇒ periodic bdry condition.

$$Z = \int_{+i\epsilon} \mathcal{D}\pi\mathcal{D}\phi\; e^{i\int dt \int d^d x\,(\pi\dot{\phi} - \mathcal{H})}$$ for any QFT's

are also called partion function.

Partition functions are functions of coupling constants, mass parameters and the boundary conditions (or the temperature); hard to compute, though.

Partition function is a useful concept, because it can be regarded as a generating function of correlators of observables:

$$Z[g, m; h_I] = \int \mathcal{D}\pi\mathcal{D}\phi\; e^{i\int dt \int d^d\vec{x}\,\{(\pi\dot{\phi} - \mathcal{H}) + h_I(x)\mathcal{O}_I(x)\}}$$

$$\langle \mathcal{O}_I(x)\rangle = \left(\frac{1}{Z}\frac{\partial}{i\,\partial h_I(x)} Z\right),$$ which is a function of $g, m; \langle h_I\rangle$. (and bdry cond.)

Its first derivative at the $\langle h_I\rangle = 0$ limit (susceptibility) is given by

$$\boxed{\frac{\partial^2}{\partial(ih_I(x))\,\partial(ih_J(y))}\ln Z = \langle\Omega|T\{\mathcal{O}_I(x)\,\mathcal{O}_J(y)\}|\Omega\rangle_{\langle h\rangle=0} - \langle\Omega|\mathcal{O}_I(x)|\Omega\rangle\langle\Omega|\mathcal{O}_J(y)|\Omega\rangle_{\langle h\rangle=0}}$$