§6.4 Partition function, free energy, effective action (formal)

In statistical mechanics,

$$Z := \sum_{\text{all states}} e^{-\beta E_i} = \mathrm{tr}_{\mathcal{H}}\left(e^{-\beta H}\right)$$ is called the partition function of a given system.

When $H = \int d^d\vec{x}\, \mathcal{H}(\phi, \pi)$ is given,

$$\boxed{Z(\beta) = \int_{\substack{\phi(\vec{x}, t=-i\beta) \\ = \phi(\vec{x}, t=0)}} \mathcal{D}\pi\mathcal{D}\phi\ e^{i\int_0^{-i\beta} dt \int d^d\vec{x}\, (\pi\dot{\phi} - \mathcal{H})} = \int_{\substack{\phi(\vec{x}, \tau=\beta) \\ = \phi(\vec{x}, \tau=0)}} \mathcal{D}\pi\mathcal{D}\phi\ e^{-\int_0^{\beta} d\tau \int d^d\vec{x}\, (\mathcal{H} - i\pi(\partial_\tau\phi))} \quad ((t = -i\tau))}$$

trace $\Rightarrow$ periodic bdry condition.

$$Z = \int_{+i\epsilon} \mathcal{D}\pi\mathcal{D}\phi\ e^{i\int dt \int d^d x (\pi\dot{\phi} - \mathcal{H})}$$ for any QFT's

are also called partion function.

Partition functions are functions of coupling constants, mass parameters and the boundary conditions (or the temperature); hard to compute, though.

Partition function is a useful concept, because it can be regarded as a generating function of correlators of observables:

$$\boxed{Z[\beta; g, m; h_I(\vec{x})] := \int \mathcal{D}\pi\mathcal{D}\phi\ e^{-\int_0^\beta d\tau \int d^d\vec{x} \{-i\pi(\partial_\tau\phi) + \mathcal{H} - h_I(\vec{x}) \mathcal{O}_I(\vec{x})\}}}$$

- $$\langle \mathcal{O}_I(\vec{x}) \rangle^{h=0}_{\text{th.eq}} = \frac{1}{\beta} \frac{\partial}{\partial h_I(\vec{x})} \ln\left(Z[\beta; g, m; h_I]\right)\Big|_{h_I=0}$$
- $$\underbrace{\frac{\partial}{\partial h_J(\vec{y})} \left(\langle \mathcal{O}_I(\vec{x}) \rangle^{h}_{\text{th.eq}}\right)\Big|_{h=0}}_{\text{susceptibility}} = \frac{1}{\beta} \frac{\partial^2}{\partial h_J(\vec{y}) \partial h_I(\vec{x})} \ln\left(Z[\beta; g, m; h_I]\right)\Big|_{h=0}$$
$$= \beta\left(\langle \mathcal{O}_J(\vec{y}) \mathcal{O}_I(\vec{x}) \rangle^{h=0}_{\text{th.eq}} - \langle \mathcal{O}_I(\vec{x}) \rangle^{h=0}_{\text{th.eq}} \langle \mathcal{O}_J(\vec{y}) \rangle^{h=0}_{\text{th.eq}}\right)$$

$\boxed{\text{Linear response relation: susceptibility is given by a 2-pt correlation}}$

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Statistical mechanics

$$Z = \mathrm{Tr}\left[e^{-\beta H}\right] \sim e^{-\frac{E}{T} + \ln(\#(\text{states}(E)))} = e^{-\frac{(E-TS)}{T}} = e^{-\beta F}.$$

So, $Z[\beta; g, m; h_I(\vec{x})]$ $= e^{-\beta F[\beta; g, m; h_I(\vec{x})]}$
given in path-integral

$$F = \int d^d\vec{x}\ \mathcal{F}(\vec{x})$$
$\hookrightarrow$ free-energy density.

$$S = -\frac{\partial}{\partial T} F[T; \cdots] = \frac{\partial}{\partial T} T \ln\left(Z[T; g, m; h_I(\vec{x})]\right)$$
$\hookrightarrow$ entropy of a given system in the thermal equilibrium.

$\Rightarrow$ compute $Z[\beta; g, m; h_I(\vec{x})]$ to get free-energy.

Effective action (a generalization of Ginzburg-Landau theory)

since $\langle \mathcal{O}_I(\vec{x})\rangle^{h}_{th.eq} = -\frac{\partial}{\partial h_I(\vec{x})} F[h_I(\vec{x})]$ $(\beta, g, m$ implicit$)$

$$-\Gamma_{th.eq}[\langle \mathcal{O}_I\rangle_{th.eq}] := F - \int d^d\vec{x}\ h_I(\vec{x})\left(\frac{\partial F}{\partial h_I(\vec{x})}\right) = F + \int d^d\vec{x}\ h_I(\vec{x}) \langle \mathcal{O}_I(\vec{x})\rangle_{th.eq}$$

is the Legendre transform of F w.r.t. h_I.

$\Gamma_{th.eq}$: called effective action.
is a functional of $\beta; g, m; \langle \mathcal{O}_I(\vec{x})\rangle$.

◎ Think of the derivative expansion of $\Gamma_{th.eq}$
(terms involving $\langle \mathcal{O}_I(\vec{x})\rangle$, $\vec{\partial}_x \langle \mathcal{O}_I(\vec{x})\rangle$, $\vec{\partial}_x \vec{\partial}_x \langle \mathcal{O}_I(\vec{x})\rangle$ ......)

The non-derivative terms are written in the form of

$$\Gamma_{th.eq} \supset -\int d^d\vec{x}\ V_{th.eq.eff}(\langle \mathcal{O}_I(\vec{x})\rangle)$$
$\hookrightarrow$ called effective potential.

◎ Solutions to
$$\frac{\partial \Gamma_{th.eq}}{\partial \langle \mathcal{O}_I(\vec{x})\rangle} = 0 \sim \frac{\partial V_{th.eq.eff}}{\partial \langle \mathcal{O}_I(\vec{x})\rangle} = 0$$
claim ⇑ Landau-Ginzburg potential
may be $\vec{x}$-dependent $\langle \mathcal{O}_I(\vec{x})\rangle$. [eg. Peierls transition]

$\boxed{\text{The } T \to 0 \quad (\beta \to +\infty) \text{ limit:}}$

$Z[\beta \to +\infty ; g, m]_{(\text{th.eq.})}$ should be the same as

$$Z[g, m]_{+i\varepsilon} = \int_{+i\varepsilon} \mathcal{D}\pi \mathcal{D}\phi \; e^{i\int dt \int d^d\vec{x} (\pi \dot{\phi} - \mathcal{H})}.$$

So, we define

$$Z[g, m ; h_I(x^\mu)] = \int_{+i\varepsilon} \mathcal{D}\pi \mathcal{D}\phi \; e^{i\int dt \int d^d\vec{x} \left(\pi \dot{\phi} - \mathcal{H} + h_I(x) \mathcal{O}_I(x)\right)}.$$

(now, we allow $x^\mu \in \mathbb{R}^{1,d}$ dependence)

$$Z[g, m ; h_I] =: e^{-i\int dt\, d^d\vec{x}\; \mathcal{F}[g, m ; h_I(x)]},$$

so we have $\langle \Omega | \mathcal{O}_I(x) | \Omega \rangle = -i \frac{\partial}{\partial h_I(x)} \ln(Z) = -\frac{\partial}{\partial h_I(x)} \int dt\, F$.

The Legendre transform — the effective action — is

$$-\Gamma_{vac}[g, m ; \langle \mathcal{O}_I \rangle_{vac}] = \int dt\, F - \int dt\, d^d\vec{x}\; h_I(t, \vec{x}) \frac{\partial}{\partial h_I(t, \vec{x})} \left[\int dt\, F\right]$$

$$= \int dt\, F + \int dt\, d^d\vec{x}\; h_I(x) \langle \mathcal{O}_I(x) \rangle_{vac}.$$

Γ_{vac} for t-indep $\langle \mathcal{O}_I \rangle_{vac}$

$\rightarrow$ get rid of $\int dt$ from Γ_{vac}.

$\Rightarrow$ should be equal to $\Gamma_{th.eq.}\big|_{\substack{\beta \to +\infty \\ (T \to 0)}}$.

Γ_{vac}, $\Gamma_{th.eq.}$, $V_{th.eq.eff}$ as the GL potential

To see this, just note that

$$\frac{\partial \Gamma_{vac}}{\partial \langle O_I(x)\rangle_{vac}} = -h_I(x) \qquad \left[\frac{\partial \Gamma_{th.eq.}}{\partial \langle O_I(\vec{x})\rangle_{th.eq}} = -h_I(\vec{x}) \; , \quad \frac{\partial V_{th.eq.eff}}{\partial \langle O_I\rangle} = -h_I \right]$$

no $\vec{x}$-dep here

(a general property of Legendre transformation)

Your theory corresponds to $h_I(x)=0$ $\Rightarrow$ $\langle O_I(x)\rangle_{vac}$ is a solution to $\partial\Gamma_{vac}/\partial\langle O_I(x)\rangle = 0$.

$\langle O_I\rangle$ is a solution to $\partial V_{th.eq.eff}/\partial\langle O_I\rangle = 0$.

(minimum of $V_{th.eq.eff}$)

How to compute? (still formal)

☆ free energy : just think of $h_I(x)$ as a part of your theory and sum up vacuum bubble graphs.

$$\left[e^{-i\int dt\, F[h]} \propto \int \mathcal{D}\phi \; e^{i\int d^{d+1}x\,(\mathcal{L}+h_I\mathcal{O}_I)} \right]$$

split $\phi(x) \to \phi_{fl}(x) + \phi_*(x)$ so that $\left\{ \begin{array}{l} \langle\phi(x)\rangle^h = \phi_* \\ \langle \phi_{fl}(x)\rangle^h = 0 \end{array}\right\}$.

[$\phi_*(x)$ is determined by $h(x)$]

Replace $\mathcal{D}\phi(x)$ by $(\mathcal{D}\phi_{fl}(x))$, expand $(\mathcal{L}+h_I\mathcal{O}_I)$ by powers of ϕ_{fl}'s

$$(\mathcal{L}+h_I\mathcal{O}_I) = (\mathcal{L}+h_I\mathcal{O}_I)\Big|_{\phi=\phi_*} + \phi_{fl}\left[\frac{\partial}{\partial\phi}(\mathcal{L}+h_I\mathcal{O}_I)\right]\Big|_{\phi=\phi_*} + \frac{1}{2}\phi_{fl}^2\left[\frac{\partial^2}{\partial\phi^2}(\mathcal{L}+h_I\mathcal{O}_I)\right] + \cdots$$

to obtain

$$\boxed{-i\int dt\, F = (\text{const}) + i\int d^{d+1}x\,(\mathcal{L}+h_I\mathcal{O}_I)\Big|_{\phi=\phi_*(h)} + \left(\begin{array}{c}\text{vac. bubble graphs of } \phi_{fl}\text{'s} \\ \text{that are fully connected}\end{array}\right)}$$

☆ we can ignore the $\phi_{fl}\left[\frac{\partial}{\partial\phi}(\mathcal{L}+h_I\mathcal{O}_I)\right]_{\phi=\phi_*}$ term in computing the vac. bubbles (by definition)

$$\left(\overset{\phi_{fl}}{\text{- - - -}}\circ\ \frac{\partial}{\partial\phi}[\mathcal{L}+h_I\mathcal{O}_I]\Big|_{\phi=\phi_*} + \overset{\phi_{fl}}{\text{- - - -}}\,(\text{any graphs}) = 0 \right)$$

☆ To write down F in terms of $h_I(x)$, we need to work out the $\phi_* - h$ relation — however.

Suppose that $\sum_i h_i\phi_i$ for all the elementary fields (→ coordinates of path integral in your effective theory) is contained in $\sum_I h_I\mathcal{O}_I$..

Then $\langle\phi_i(x)\rangle = -\frac{\partial}{\partial h_i(x)}\int F\,dt = \langle(\phi_{i,fl}+\phi_{i,*})(x)\rangle = \langle\phi_{i,*}(x)\rangle$

☆ Effective action

☆☆ simpler cases [only elementary fields (in your eff. theory) get non-0 expectation values.]

examples (the Ising model (scalar ϕ^4 theory)
the Higgs doublet in the Standard Model.
an inflaton.)

Retain $\sum_i h_i \phi_i$ and drop all other terms in $\sum_I h_I \mathcal{O}_I$.

Do the same computation as for the free energy.

except $\left(\mathcal{L} + \sum_i h_i \phi_i\right)\Big|_{\phi_i = \phi_{i*}} \Longrightarrow \left(\mathcal{L} + \sum_i h_i \phi_i\right)\Big|_{\phi_i = \phi_{i*}} - \sum_i h_i \langle \underset{\overset{\shortparallel}{\phi_{i*}}}{\phi_i} \rangle = \mathcal{L}\Big|_{\phi_i = \langle \phi_i \rangle}$

$$\boxed{i\Gamma_{vac}[\langle\phi\rangle] = i\int d^{d+1}x\, [\mathcal{L}]\Big|_{\phi=\langle\phi\rangle} + \underbrace{\left(\text{vac. bubbles fully connected}\right)}_{(\text{in } \phi_{fe}.)}}$$

quantum correction to the action.

☆☆ super conductivity

$\mathcal{L}_0 = \psi^\dagger (i\partial_t - \mathcal{H}_0)\psi$ $\quad$ ψ: 2-component spinor field.

$\mathcal{L}_{int} = \frac{1}{\Lambda^2}(\psi^\dagger\psi)(\psi^\dagger\psi)$ $\quad \Leftarrow$ generated by phonon longitudinal mode exchange.

$$Z = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \; e^{i\int dt \int d\vec{x}\, (\mathcal{L}_0 + \mathcal{L}_{int})}$$

$$\propto \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\sigma\mathcal{D}\bar{\sigma} \; e^{i\int dx\, \mathcal{L}_0 + (-\Lambda^2|\sigma|^2 + \sigma\psi\psi + \sigma^*\psi^\dagger\psi^\dagger)}$$

$\Longrightarrow \Gamma[\langle\sigma\rangle] \supset -\int d^{d+1}x\; V_{eff}(\langle\sigma\rangle)$ is the Ginzburg–Landau potential.

$\langle\sigma\rangle$ at the minimum corresponds to $\frac{\langle\psi^\dagger\psi^\dagger\rangle}{\Lambda^2}$.