Correlation functions in a canonical ensemble

$$\langle \mathcal{O}_I(\vec{x})\mathcal{O}_J(\vec{y})\rangle_\beta - \langle \mathcal{O}_I(\vec{x})\rangle_\beta \langle \mathcal{O}_J(\vec{y})\rangle_\beta = T^2 \frac{\partial^2}{\partial h_I(\vec{x})\partial h_J(\vec{y})}\left(\ln Z\right) = -T\frac{\partial^2}{\partial h^2}(F_{th})$$

$$\parallel$$

$$T\frac{\partial}{\partial h_I(\vec{x})}\langle \mathcal{O}_J(\vec{y})\rangle_{th}$$

How is this related to the "effective action" $\Gamma_{th.eq}$?

Remember that

$$-\frac{\partial \Gamma_{th.eq}}{\partial \langle \mathcal{O}_I(\vec{x})\rangle_{th}} = \frac{\partial}{\partial \langle \mathcal{O}_I(\vec{x})\rangle_{th}}\left(F + \int d^d\vec{x}\, h_I(\vec{x})\langle \mathcal{O}_I(\vec{x})\rangle\right)$$

$$= \frac{\partial h_K(\vec{z})}{\partial\langle \mathcal{O}_I(\vec{x})\rangle_{th}}\underbrace{\left(\frac{\partial F}{\partial h_K(\vec{z})} + \langle \mathcal{O}_K(\vec{z})\rangle_{th}\right)}_{=0} + h_I(\vec{x}) = h_I(\vec{x}),$$

we find that

$$-\beta\frac{\partial^2 \Gamma_{th.eq}}{\partial\langle \mathcal{O}_K(\vec{w})\rangle_{th}\,\partial\langle \mathcal{O}_I(\vec{x})\rangle} = \beta\frac{\partial}{\partial\langle \mathcal{O}_K(\vec{w})\rangle_{th}}\left(h_I(\vec{x})\right)$$

is the inverse matrix of $T\frac{\partial}{\partial h_I(\vec{x})}\langle \mathcal{O}_J(\vec{y})\rangle_{th}$.

So, if $\Gamma_{th.eq} \sim \int d^d\vec{x}\left\{\frac{1}{2}(\vec{\nabla}\mathcal{O})(\vec{\nabla}\mathcal{O}) + \frac{1}{2}m_{eff}^2(\mathcal{O})^2\right\} + \left(\text{higher order in } \mathcal{O}\right)$

then $$\frac{\partial^2\Gamma_{th.eq}}{\partial\langle \mathcal{O}(-\vec{q})\rangle\,\partial\langle \mathcal{O}(\vec{p})\rangle} = (2\pi)^d\delta^d(\vec{p}-\vec{q})\left(p^2+m_{eff}^2\right).$$

Taking its inverse,

$$\langle \mathcal{O}(\vec{x})\,\mathcal{O}(\vec{y})\rangle = \int\frac{d^d\vec{p}}{(2\pi)^d}\frac{e^{i\vec{p}\cdot(\vec{x}-\vec{y})}}{(p^2+m_{eff}^2)} \sim \frac{e^{-m_{eff}|\vec{x}-\vec{y}|}}{|\vec{x}-\vec{y}|^{d-2}}.$$

$1/m_{eff}(\beta;g,m)$ is the correlation length.

Critical temperature $\Leftrightarrow$ $\beta_c = 1/T_c$ s.t. $m_{eff}^2(\beta_c;g,m) = 0$.

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§7.5 Thermal field theory (imaginary time formalism)

How can we compute F or $\Gamma_{\text{th eq}}$ as a function of β, g, m?

Consider cases where there is no explicit time-dependence in H_0 or H.

Then... $U(t) := e^{iH_0 t} e^{-iHt}$ in Minkowski space.

$$\Longrightarrow \quad U(t) = T\left(\exp\left[-i\int_0^t dt' \, V_I(t')\right]\right)$$

interaction picture

Similarly $U_{th}(\tau) := e^{H_0\tau} e^{-H\tau}$

$$\Longrightarrow \quad U_{th}(\tau) = T\left(\exp\left[-\int_0^\tau d\tau' \, V_I(\tau')\right]\right).$$

Now the free energy / "effective action" can be computed through

$$\mathrm{Tr}\left[e^{-\beta H}\right] = \mathrm{Tr}\left[U_{th}(\beta)\, e^{-\beta H_0}\right]$$

$$= \frac{\mathrm{Tr}\left[T\left(\exp\left[-\int_0^\beta d\tau' V_I(\tau')\right]\right) e^{-\beta H_0}\right]}{\mathrm{Tr}\left[e^{-\beta H_0}\right]} \times \mathrm{Tr}\left[e^{-\beta H_0}\right].$$

The second factor: a partition function of a free theory.

Perturbative computation of the first factor: $(t \to -i\tau)$

$$\frac{\mathrm{Tr}\left[T\{\phi_I(\tau,\vec{x})\,\phi_I(0,\vec{0})\}\, e^{-\beta H_0}\right]}{\mathrm{Tr}\left[e^{-\beta H_0}\right]} \quad \text{for } \phi_I(\tau,\vec{x}) = \int \frac{d^d\vec{p}}{(2\pi)^d} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x} - E_{\vec{p}}\tau} + a_{\vec{p}}^\dagger\, e^{-i\vec{p}\cdot\vec{x} + E_{\vec{p}}\tau} \right)$$

$$\begin{pmatrix} \text{cf.} \\ \langle n|a^\dagger a|n\rangle = n \\ \langle n|a a^\dagger|n\rangle = (n+1) \end{pmatrix}$$

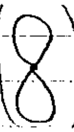
is $\displaystyle\int \frac{d^d\vec{p}}{(2\pi)^d} \frac{1}{(2E_{\vec{p}})} \left\{ e^{i\vec{p}\cdot\vec{x} - E_{\vec{p}}\tau} (n_{E_{\vec{p}}} + 1) + e^{-i\vec{p}\cdot\vec{x} + E_{\vec{p}}\tau} (n_{E_{\vec{p}}}) \right\} =: \Delta_F(\tau)$

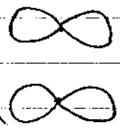
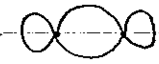
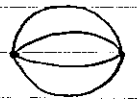
$$n_{E_{\vec{p}}} := \frac{1}{(e^{\beta E_{\vec{p}}} - 1)} \quad \text{(Bose-Einstein distribution)}$$

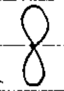
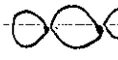
- $\Delta_F(\tau,\vec{x})$ contains the vacuum piece $\Delta(t,\vec{x})\big|_{t \to -i\tau}$.
- $\displaystyle \Delta_F = \int \frac{d^d\vec{p}}{(2\pi)^d} \frac{1}{2E_{\vec{p}}} \left\{ e^{i\vec{p}\cdot\vec{x}} \left(\sum_{n=0}^{\infty} e^{-E_{\vec{p}}(\tau + n\beta)} \right) + e^{-i\vec{p}\cdot\vec{x}} \left(\sum_{m=1}^{\infty} e^{E_{\vec{p}}(\tau - m\beta)} \right) \right\}$

multiple winding propagation in the Euclidean time direction.

Take ϕ^4-theory as an example.

$$\frac{\mathrm{Tr}\left[T\left(\exp\left[-\int_0^\beta d\tau' V_I(\tau')\right]\right)e^{-\beta H_0}\right]}{\mathrm{Tr}\left[e^{-\beta H_0}\right]} = 1 + \lambda\Big($$  $$\Big)$$

$$+\lambda^2\Big($$  $$+$$  $$+$$  $$\Big) + \cdots$$

$$= \exp\Big[\lambda\Big($$  $$\Big) + \lambda^2\Big($$  $$+$$  $$\Big) + O(\lambda^3)\Big]$$

fully connected diagrams only.

* The thermal propagator is used for this computation.
  $\Rightarrow$ The free energy / "effective action" depends on β, λ etc.

* $V_{\mathrm{th.ef.eff}} \longrightarrow V_{\mathrm{vac.eff}}$ in the $\beta \to +\infty$ limit.
  $V_{\mathrm{vac.eff}}$ is still different from V_{vac} in $-\mathcal{L}$.
  due to quantum corrections.

The second factor $\mathrm{Tr}[e^{-\beta H_0}]$

(the free (bilinear) part expanded around $\langle\phi\rangle$)

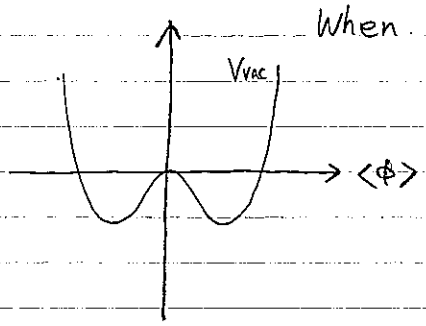
☆ A free scalar boson = harmonic oscillators labeled by $\vec{k} \in \mathbb{R}^d$.

◎ $$Z = \prod_{\vec{k}} \left(\sum_{n_k=0}^{\infty} e^{-\beta E(n_k+\frac{1}{2})} \right) = \prod_{\vec{k}} \frac{1}{(e^{+\beta E/2} - e^{-\beta E/2})} = \exp\left[-V_d \int \frac{d^d k}{(2\pi)^d} \ln\left(e^{\beta E/2} - e^{-\beta E/2}\right) \right]$$

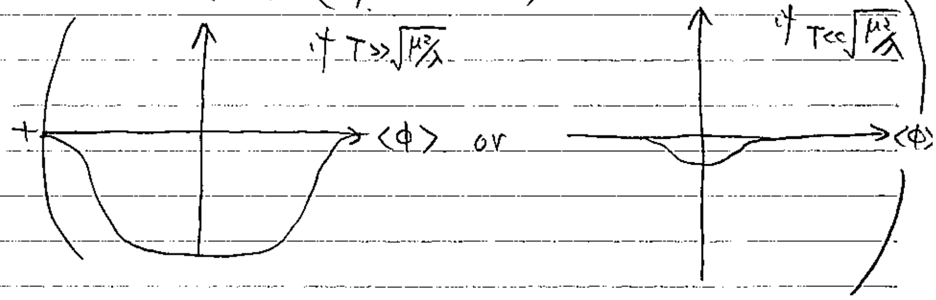
so $$F = T \int \frac{d^d \vec{k}}{(2\pi)^d} \ln\left(e^{\beta E/2} - e^{-\beta E/2}\right) = \int \frac{d^d k}{(2\pi)^d} \left\{ \underbrace{\frac{E_{\vec{k}}}{2}}_{\substack{\text{quantum} \\ \text{(Casimir)} \\ \text{divergent}}} + \underbrace{T \ln\left(1 - e^{-\beta E_{\vec{k}}}\right)}_{\substack{\text{thermal contrib.} \\ \text{finite}}} \right\}$$

The thermal contrib. $\Rightarrow$ $\begin{cases} \text{if } T \gg m(\langle\phi\rangle) & \Delta F \sim -O(1) \times T^{d+1} \\ \text{if } T \ll m(\langle\phi\rangle) & \Delta F \sim -O(1) \cdot m^d T e^{-\beta m} \end{cases}$

When $V_{vac.}(\phi) = -\frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4$



$m^2(\langle\phi\rangle) = \left(-\mu^2 + \lambda \langle\phi\rangle^2\right)$



F changes as β varies.

◎ $$Z \propto \int \mathcal{D}\phi \; e^{-\int_0^\beta d\tau \int d^d x \frac{1}{2} \phi(-\nabla^2 + m^2(\langle\phi\rangle))\phi} = e^{-V_d \int \frac{d^d k}{(2\pi)^d} \sum_n \frac{1}{2} \ln\left(E_k^2 + (2\pi T n)^2\right)}$$

Gaussian integral

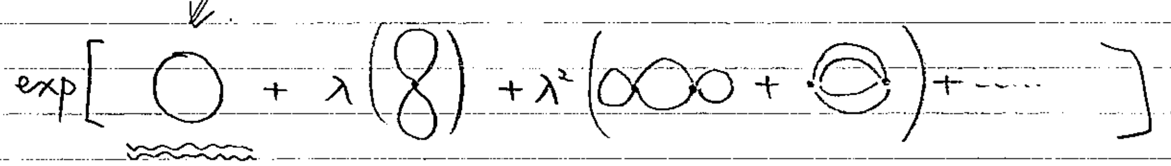
Mathematically both

$$\prod_{n \in \mathbb{Z}} \frac{T^2}{[E^2 + (2\pi T n)^2]} \quad \text{and} \quad \frac{1}{(e^{\beta E/2} - e^{-\beta E/2})^2} \quad \text{as functions of } (\beta E/2)$$

have a pole at $\beta E = \pi i n$ of order 2 $\sim \frac{(n\text{-indep.})}{(\beta E/2 - \pi i n)^2} + \cdots$

So they are proportional. [(Z for a free theory) $= \mathrm{Tr}[e^{-\beta H_0}]$ is called functional determinant.]

◎ The functional determinant corresponds to
(the second term)



reasonings for this claim

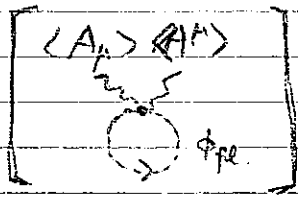
- $\frac{\partial \Gamma_{th}}{\partial \langle \phi \rangle}$ = ⊗---○ = first derivative of ○ in the $\langle \phi \rangle$ background
- opposite sign for bosons and fermions.

Thermal mass (Debye / Thomas-Fermi screening)

Look at the quadratic term of $\Gamma_{th\,eq}$ in a theory $\mathcal{L} \supset (D_\mu \phi)^\dagger (D^\mu \phi)$

$D_\mu = (\partial_\mu - ieQA_\mu)$

There is a term in $\Gamma_{th\,eq}$ from



with the thermal mass.

$d\omega$

$$(eQ)^2 \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \left\{ (1 + n_{E_p}) + n_{E_p} \right\} \sim e^2 Q^2 T^2 \quad \text{if } m \ll T.$$

ignore this quantum div. contrib.

$\boxed{\text{Non-zero mass} \Longrightarrow \text{finite correlation length.}}$

(screening)