§7.6 Effective theory and coarse graining

When a system (theory, model) has both heavy partiles and light particles (ϕ and ψ), it is possible to integrate out ϕ at the very beginning, if you are interested only in correlation functions involving ψ's, (not ϕ's).

$$Z[\chi_H, \chi_L] := \int \mathcal{D}\phi\, \mathcal{D}\psi\; e^{i\int d^{d+1}x \left[\left(\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}M^2\phi^2\right) + \frac{1}{2}(\partial_\mu\psi)^2 - g\phi\psi\psi + \phi\chi_H + \psi\chi_L\right]}$$

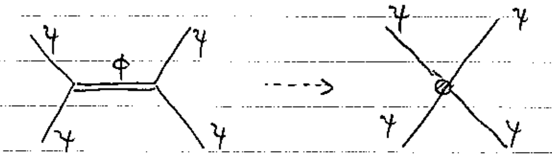
external fields (so Z is a generating functional) of all correlation fcn's

☆ complete a square in the exponent

☆ carry out the Gaussian integral.

$$\Longrightarrow Z[\chi_H, \chi_L] \propto \int \mathcal{D}\psi\; e^{i\int d^{d+1}x \left[\frac{1}{2}(\partial_\mu\psi)^2 + \psi\chi_L + (g\psi\psi - \chi_H)\frac{1}{\partial^2 + m^2}(g\psi\psi - \chi_H)\right]}$$

to get the action of the low-energy effective theory.



by "integrating out" the heavy field ϕ.

Also possible (and useful) to integrate out

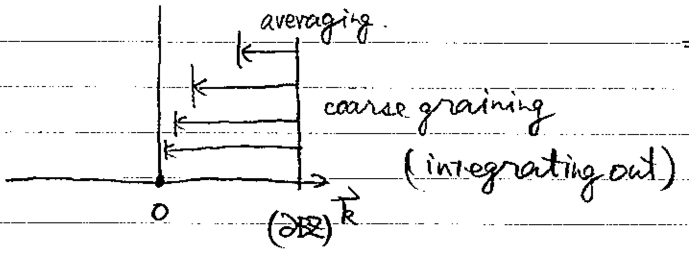
- only two components [(−1)-eg space of γ^0] in the four component Dirac fermion
- only negative frequency modes of a complex scalar

so we obtain a non-relativistic effective theory of $\left\{\begin{array}{l} e^- \\ \text{cold atom} \end{array}\right\}$

The idea dates back to Born–Oppenheimer approximation.

Consider a spin system turned into a field theory (eg. ϕ^4 theory)

- spacial momentum : $|\vec{k}| \leq$ bdry of the Brillouin zone. $\sim \pi/a$
- scalar field(s) : local average of magnetization
  (already a length scale of smearing (averaging) is in place) $> a$.
- coarse graining (粗視化) / block spin transformation.
  $\Leftrightarrow$ integrating out $\phi(\vec{k})$'s with larger $\vec{k}$'s



$\Rightarrow$ In the effective action,
new terms may show up, and coupling constants may be modified accordingly.

- coupling constants : fcns of $|\vec{k}|_{max}$ $\quad g(|\vec{k}|_{max})$
- free energy / effective action (GL potential) : depend on $g(|\vec{k}|_{max})$ & $|\vec{k}|_{max}$ apparently
  but should not dep. on $|\vec{k}|_{max}$ in the end

$\frac{\partial g(|\vec{k}|_{max})}{\partial \ln(|\vec{k}|_{max})}$ is related to the $|\vec{k}|_{max}$-dependence of the computation of the effective action.

$\boxed{\text{renormalization (group)}}$

§ 7.7 Brief introduction to $\left\{\begin{matrix}\text{In-In} \\ \text{Schwinger-Keldysh} \\ \text{real time}\end{matrix}\right\}$ formalism

Consider a system with

$$\boxed{\begin{aligned} H &= H_0 + H_{int} \\ \tilde{H} &= H + H' \end{aligned}}$$

H_{int} as in QED...

H' : coupling with time-varying background fields)

Start from a system in an equilibrium under the Hamiltonian H.

H' is turned on after time t_0.

(· AC electric field (conductance)
· magnetic field switched $\left\{\begin{matrix}\text{on}\\\text{off}\end{matrix}\right\}$ (NMR)
· time-varying $\left\{\begin{matrix}\text{inflaton}\\\text{metric}\\\text{order parameter}\end{matrix}\right\}$)

so the system deviates from the original equilibrium.

What happens then ?

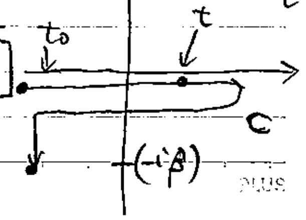
The easiest question to ask : $\dfrac{\mathrm{Tr}[\mathcal{O}(t)\rho]}{\mathrm{Tr}[\rho]}$

$$\rho = e^{-\beta H} = e^{-\beta H_0}\, P\left[\exp\left(-\int_0^\beta d\tau'\, H_{int\,H_0}(-i\tau + t_0)\right)\right]$$

$$\begin{aligned}
\mathcal{O}(t) &= \left(P\left[\exp\left(-i\int_{t_0}^t dt''\,\tilde{H}(t'')\right)\right]\right)^{-1} \mathcal{O}(t_0) \left(P\left[\exp\left(-i\int_{t_0}^t dt'\,\tilde{H}(t')\right)\right]\right) \\
&= \left(\qquad \right)^{-1} P\left[\exp\left(-i\int_{t_0}^t dt'''\, H_0\right)\right] \left(P\left[\exp\left(-i\int_{t_0}^t dt''\, H_0\right)\right]\right)^{-1} \mathcal{O}(t_0) \left(P\left[\exp\left(-i\int_{t_0}^t H_0\right)\right]\right) \\
&\qquad \left(P\left[\exp\left(-i\int_{t_0}^t H_0\right)\right]\right)^{-1} \cdot P\left[\exp\left(-i\int_{t_0}^t dt'\,\tilde{H}(t')\right)\right] \\
&= P\left[\exp\left(-i\int_{t_0}^t dt''\,(H' + H_{int})_{H_0}(t'')\right)\right]^{-1} \cdot \mathcal{O}_{H_0}(t) \cdot P\left[\exp\left(-i\int_{t_0}^t dt'\,(H' + H_{int})_{H_0}(t')\right)\right].
\end{aligned}$$

using the interaction picture operators.

$$\Longrightarrow \frac{\mathrm{Tr}[\mathcal{O}(t)\rho]}{\mathrm{Tr}[\rho]} = \frac{\mathrm{Tr}\left[e^{-\beta H_0} \cdot P\left[\mathcal{O}_I(t)\, \exp\left(-i\int_C dt'\,(H' + H_{int})_I(t')\right)\right]\right]}{\mathrm{Tr}\left[e^{-\beta H_0}\, P\left[\exp\left(-i\int_C dt'\,(H' + H_{int})_I(t')\right)\right]\right]}$$



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an example : AC electric field on a metal

$$\vec{\mathcal{O}} = \left(\psi^\dagger \frac{-i\vec{\partial}}{m} \psi\right)$$ ← spacial component of the Noether current.

2 component fermion

$$H' = \psi^\dagger \frac{(-ieQ)}{2m}(\vec{A}\cdot\vec{\partial} + \vec{\partial}\cdot\vec{A})\psi \qquad \vec{A} = \frac{\vec{E}_0}{i\omega} e^{-i\omega t}$$ classical background turned on after t_0

The AC conductance is defined by $\underline{\langle \vec{\mathcal{O}} \rangle = \sigma \vec{E}_0}$ (linear response)

$$\left\langle \frac{1}{\omega} \int^t dt' \left[\left(\psi^\dagger \frac{-i\vec{\partial}}{m}\psi\right)(t), \left(\psi^\dagger \frac{-i\vec{\partial}}{m}\psi\right)(t')\right] e^{-i\omega t'} \right\rangle_\beta \sim \sigma$$

$$\psi_1 \sim e^{-i(E(\vec{k})-\mu)t} a_{\vec{k}} + e^{i(\mu - E(\vec{k}))t} b_{\vec{k}}^\dagger$$
$$\psi_1^\dagger \sim e^{i(E(\vec{k})-\mu)t} a_{\vec{k}}^\dagger + e^{-i(\mu - E(\vec{k}))t} b_{\vec{k}}$$

(electron $(E(\vec{k}) > \mu) \Rightarrow a, a^\dagger$; hole $(E(\vec{k}) < \mu) \Rightarrow b, b^\dagger$)

t'-dep background : possible to excite $a_{\vec{k}}^\dagger b_{-\vec{k}}^\dagger$ (tilt the Fermi surface)

$\boxed{\text{This is not a time-ordered correlation function, but a contour-ordered correlation function.}}$

A case study

$$\mathrm{Tr}\left(e^{-\beta H_0}\left[(\psi^\dagger\psi)(t,\vec{x}), (\psi^\dagger\psi)(0,\vec{y})\right]\right) \quad \text{with } t > 0$$ (forget about $\frac{-i\vec{\partial}}{m}$ etc. not essential)

Note that

$$\Theta(t)\left[\psi_1^\dagger\psi_1, \psi_2^\dagger\psi_2\right] = \Theta(t)\left(\psi_1^\dagger\{\psi_1, \psi_2^\dagger\}\psi_2 - \psi_1^\dagger\psi_2^\dagger\psi_1\psi_2 - \psi_2^\dagger\{\psi_2, \psi_1^\dagger\}\psi_1 + \psi_2^\dagger\psi_1^\dagger\psi_2\psi_1\right)$$
$$= \Theta(t)\left(\{\psi_1, \psi_2^\dagger\}\psi_1^\dagger\psi_2 - \{\psi_2, \psi_1^\dagger\}\psi_2^\dagger\psi_1\right).$$

So, we need three propagators.

PLUS

$$\Delta^R(t,\vec{x}) := \Theta(t)\frac{\mathrm{Tr}\left[e^{-\beta H_0}\{\psi(t,\vec{x}),\psi^\dagger(0,\vec{0})\}\right]}{\mathrm{Tr}\left[e^{-\beta H_0}\right]} = \Theta(t)\langle 0|\{\psi(t,\vec{x}),\psi^\dagger(0,\vec{0})\}|0\rangle$$

$$\Delta^A(t,\vec{x}) := -\Theta(-t)\frac{\mathrm{Tr}\left[e^{-\beta H_0}\{\psi(t,\vec{x}),\psi^\dagger(0,\vec{0})\}\right]}{\mathrm{Tr}\left[e^{-\beta H_0}\right]} = -\Theta(-t)\langle 0|\{\psi(0,\vec{0}),\psi^\dagger(t,\vec{x})\}|0\rangle$$

$$\Delta^<(t,\vec{x}) := -\frac{\mathrm{Tr}\left[e^{-\beta H_0}\,\psi^\dagger(0,\vec{0})\,\psi(t,\vec{x})\right]}{\mathrm{Tr}\left[e^{-\beta H_0}\right]}$$

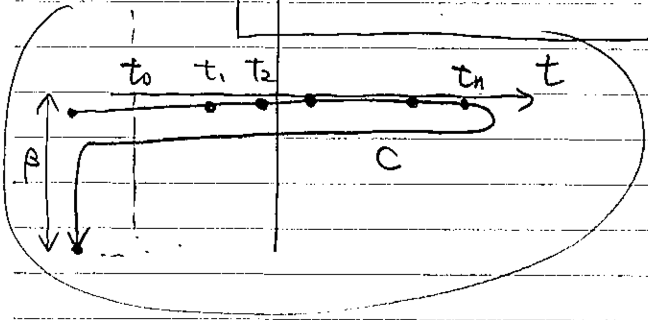
$$\Delta^>(t,\vec{x}) := \frac{\mathrm{Tr}\left[e^{-\beta H_0}\,\psi(t,\vec{x})\,\psi^\dagger(0,\vec{0})\right]}{\mathrm{Tr}\left[e^{-\beta H_0}\right]}$$

(There is one relation : $(\Delta^> - \Delta^<) = (\Delta^R - \Delta^A)$.)

$$\frac{\mathrm{Tr}\left[e^{-\beta H_0}\left[(\psi^\dagger\psi)(t,\vec{x}),(\psi^\dagger\psi)(0,\vec{y})\right]\right]}{\mathrm{Tr}\left[e^{-\beta H_0}\right]}\Theta(t) = -\Big(\Delta^R(t,\vec{x}-\vec{y})\,\Delta^<(-t,\vec{y}-\vec{x}) + \Delta^<(t,\vec{x}-\vec{y})\,\Delta^A(-t,\vec{y}-\vec{x})\Big).$$

It is known that any of

$$\frac{\mathrm{Tr}\left[T\{\mathcal{O}_n(t_n)\cdots\mathcal{O}_1(t_1)\}\rho\right]}{\mathrm{Tr}[\rho]}$$ can be computed perturbatively by using the 3 propagators above.



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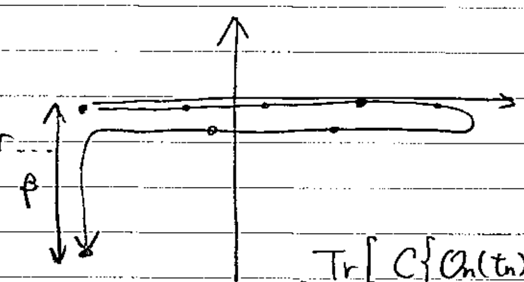

$$\frac{\mathrm{Tr}\left[C\{\mathcal{O}_n(t_n)\cdots\mathcal{O}_1(t_1)\}\rho\right]}{\mathrm{Tr}[\rho]}.$$

(of any use??)

(def) out-of-time-order(ed) correlation functions : not just going up and down just once.

like $\langle |[A(t),B(0)]|^2\rangle = \langle [B(0)^\dagger, A(t)^\dagger][A(t),B(0)]\rangle$