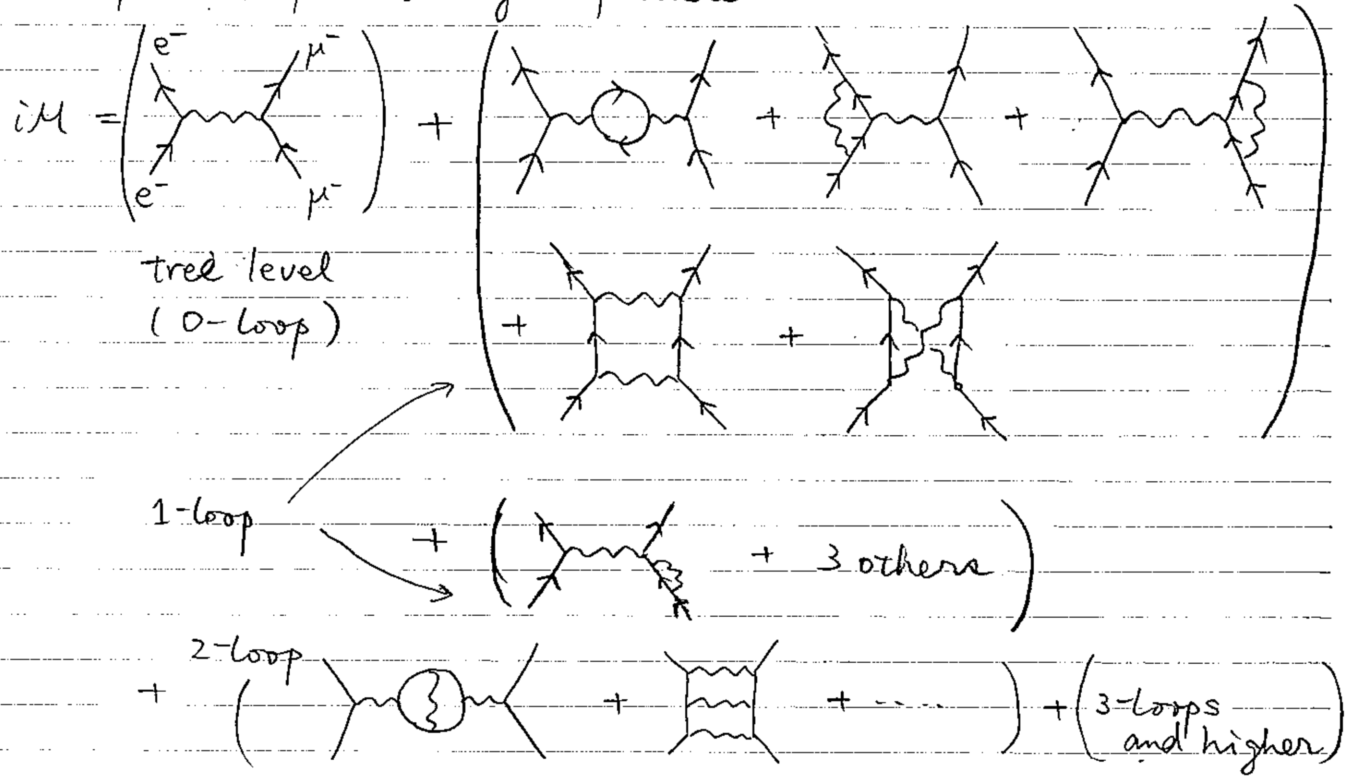
§8 Introduction to 1-loop computations

In perturbative calculations, contributions to a given correlation function / scattering amplitude are sorted out in the order of the # of loops.

ex $e^- + \mu^- \to e^- + \mu^-$ scattering amplitude.

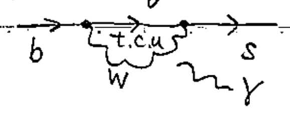


Time evolution of a state may involve creation / annihilation of e^+e^-, $\mu^+\mu^-$, γ virtually in the intermediate processes. multiple times.

Scattering amplitudes are therefore in the form of

$$iM = e^2 \times (\text{kinematics}) + e^4 \times (\text{kinematics}) + e^6 \times (\text{kinematics}) + \cdots .$$

☆ 1-loop computations (and maybe higher loop) are necessary when high precision is required.

☆ Some processes are absent at tree level, and generated at higher-loop for the first time. (eg: )

an example: anomalous magnetic moment of μ^-

The three point amplitude $(\gamma^{(*)} + \mu^- \to \mu^-)$ only has

$$iM^{\text{tree}} = i(-Qe)\,\epsilon_\mu \boxed{\bar{u}(\vec{p}\,')\,\gamma^\mu\, u(\vec{p})}$$ at the tree level.

$[p'_\mu = p_\mu + q_\mu]$

(interested in the $q^2 = 0$ limit.)

At the 1-Loop level,

$$iM^{(1\text{-loop})} = \int \bar{u}(\vec{p}\,')\,[i(-Q_\mu e)\gamma^\kappa]\, \frac{i[(\not{p}' - \not{k}) + M]}{((p'-k)^2 - M^2 + i\epsilon)}\,[i(-Q_\mu e)\gamma^\mu]\, \frac{i[(\not{p}-\not{k})+M]}{((p-k)^2 - M^2 + i\epsilon)}$$

$$[i(-Q_\mu e)\gamma^\lambda]\; u(\vec{p}) \times \left(\frac{-i\eta_{\kappa\lambda}}{k^2 + i\epsilon}\right) \times \epsilon_\mu(q)\; \frac{d^4k}{(2\pi)^4}$$

$$= (-i)\int \frac{d^4k}{(2\pi)^4}\,[i(-Qe)]\,(Qe)^2\, \frac{\bar{u}(\vec{p}\,')\,\gamma^\kappa[(\not{p}'-\not{k})+M]\,\not{\epsilon}\,[(\not{p}-\not{k})+M]\,\gamma_\kappa\, u(\vec{p})}{[k^2+i\epsilon]\,[(p'-k)^2 - M^2 + i\epsilon]\,[(p-k)^2 - M^2 + i\epsilon]}.$$

A trick that makes integration easy:

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[xA+(1-x)B]^2} \qquad (\mathrm{RHS}) = \int_0^1 dx \frac{1}{(A-B)^2\left(x+\frac{B}{A-B}\right)^2} = \frac{1}{(A-B)^2}\left(\frac{A-B}{B} - \frac{A-}{A}\right.$$

$$\frac{1}{ABC} = \int_{\triangle} dx\, dy \frac{2}{[xA+yB+(1-x-y)C]^3}$$

$$(\mathrm{RHS}) = \int_0^1 dx \int_0^x dy \frac{2}{(B-C)^3\left[y+\frac{xA+(1-x)C}{(B-C)}\right]^3} = \int_0^1 dx \frac{1}{(B-C)}\left(\frac{1}{[xA+(1-x)C]^2} - \frac{1}{[xA+(1-x)B\right.$$

$$= \frac{1}{(B-C)}\left(\frac{1}{AC} - \frac{1}{AB}\right) = \frac{1}{ABC}$$

$$\boxed{\frac{1}{A_1A_2\cdots A_n} = \int_{0\le x_i} d^n x \frac{\delta(\Sigma_i x_i - 1)(n-1)!}{[x_1A_1+x_2A_2+\cdots+x_nA_n]^n}}$$

by induction
$$(\mathrm{RHS})_n = \frac{1}{(A_{n-1}-A_n)}\left\{(\mathrm{RHS})_{n-1}(A_1,\ldots,\check{A}_{n-1},A_n) - (\mathrm{RHS})_{n-1}(A_1,\ldots,A_{n-1},\check{A}_n)\right\}$$
$$= \frac{1}{(A_{n-1}-A_n)} \times \frac{1}{(A_1A_2\cdots A_{n-2})}\left(\frac{1}{A_n} - \frac{1}{A_{n-1}}\right) = (\mathrm{LHS})_n \quad //$$

If necessary

$$\frac{1}{A^2BC} = -\frac{\partial}{\partial A}\left(\frac{1}{ABC}\right) = \int_{\triangle} dx\, dy \frac{2\cdot 3\, x}{[xA+yB+(1-x-y)C]^4}$$

etc.

So, in particular

$$\frac{1}{(k^2+i\epsilon)}\frac{1}{[(p'-k)^2-M^2+i\epsilon]}\frac{1}{[(p-k)^2-M^2+i\epsilon]}$$

$$= \int_{\triangle} dx\, dy \frac{2}{[x(p'-k)^2+y(p-k)^2+(1-x-y)k^2-(x+y)M^2+i\epsilon]^3}$$

used $(p')^2 = M^2 = p^2$ on-shell

$$= \int_{\triangle} dx\, dy \frac{2}{[k^2-2k\cdot(xp'+yp)+i\epsilon]^3}$$

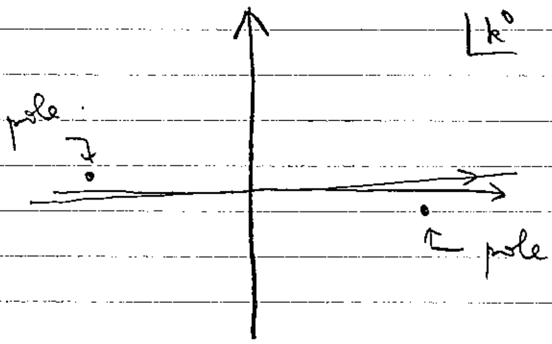
$$\left\{\begin{aligned}&(xp'+yp)^2 = x^2M^2+y^2M^2+2xy\, p'\cdot p\\ &q^2 = (p'-p)^2 = 2M^2-2p'\cdot p\end{aligned}\right.$$

$$= \int_{\triangle} dx\, dy \frac{2}{[(k')^2-((x+y)^2M^2-xyq^2)+i\epsilon]^3}$$

$k' := k-(xp'+yp)$ to complete a square.

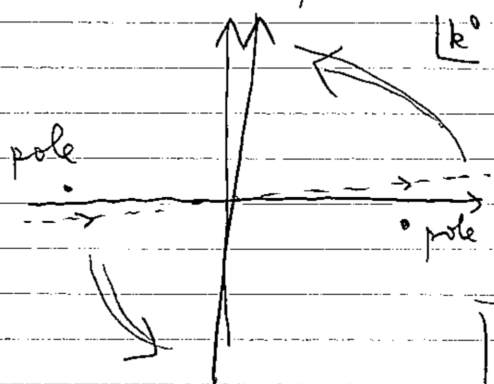
Now, we think of carrying out $\frac{d^4k}{(2\pi)^4}$ before $dx\,dy$.

The k^0-integration, in particular, has the following structure:



$$\frac{dk^0}{2\pi}\,\frac{(\text{numerator})}{\left[(k^0)^2 - \{\underset{\text{pos.}}{(\vec{k})^2} + \underset{\text{pos.}}{(x+y)^2M^2} + \underset{\text{pos.}}{xy}\,\underset{\substack{\text{pos.}\\ \text{if } q \text{ is}\\ \text{spacelike}}}{(-q^2)}\} + i\varepsilon\right]^3}$$

So, it is possible to rotate the contour of integration into



[called Wick rotation]

The new integration is parametrized by $k'^0 = i k'^4$ (so $k^4 \in \mathbb{R}$)

$$\boxed{-i\int_{-\infty}^{+\infty}\frac{dk'^0}{(2\pi)}\int_{\mathbb{R}^3}\frac{d^3k'}{(2\pi)^3} \Rightarrow +\int_{\mathbb{R}^4}\frac{d^4k'_E}{(2\pi)^4}}$$

$$\left[(k')^2 - \{(x+y)^2M^2 + xy(-q^2)\} + i\varepsilon\right]^3 \Rightarrow -\left[k_E^2 + \left\{\begin{array}{l}(x+y)^2M^2\\ +xy(-q^2)\end{array}\right\}\right]^3$$

The numerator

- $(M^2)\,\gamma^\kappa \not{\varepsilon}\,\gamma_\kappa = -2M^2 \not{\varepsilon}$ $\quad$ (use $\gamma^\kappa \not{A}\,\gamma_\kappa = -2\not{A}$)

- $M\left\{\gamma^\kappa (\not{p}'-\not{k})\not{\varepsilon}\,\gamma_\kappa + \gamma^\kappa \not{\varepsilon}\,(\not{p}-\not{k})\gamma_\kappa\right\}$

$$\left(\text{use } \gamma^\kappa \not{A}\not{B}\gamma_\kappa = 2\not{B}\gamma_\kappa A^\kappa - \not{A}\gamma^\kappa\not{B}\gamma_\kappa = 2(\not{B}\not{A}+\not{A}\not{B}) = 4(A\cdot B)\right)$$

$$= 4M\,\varepsilon_\mu (p'+p-2k)^\mu$$
$$= 4M\,\varepsilon_\mu\left((1-2x)p' + (1-2y)p - 2k'\right)^\mu \qquad \left(k'_\mu = k_\mu - (xp'+yp)_\mu\right)$$

- $\gamma^\kappa (\not{p}'-\not{k})\not{\varepsilon}(\not{p}-\not{k})\gamma_\kappa$

$$= -2\left[\not{(1-y)p - xp' - k'}\right]\not{\varepsilon}\left[\not{(1-x)p' - yp - k'}\right]. \quad \leftarrow (**)$$

use
$$\gamma^\kappa \not{A}\not{B}\not{C}\gamma_\kappa = \not{B}\not{C}\gamma_\kappa 2A^\kappa - \not{A}\not{C}\gamma_\kappa 2B^\kappa + \not{A}\not{B}\gamma^\kappa\not{C}\gamma_\kappa$$
$$= 2(\not{B}\not{C}\not{A} - \not{A}\not{C}\not{B} - \not{A}\not{B}\not{C})$$
$$= \underline{4(B\cdot C)\not{A}} - 2\not{C}\not{B}\not{A} - \underline{4\not{A}(B\cdot C)}$$

☆ drop all the terms with an odd power of k' $\quad$ (integration of an odd function $=0$)

☆ replace $k'^\rho k'^\sigma$ by $\frac{1}{\text{"}4\text{"}}\eta^{\rho\sigma}(k')^2$ $\quad$ ("4" is the spacetime dimension $\eta_{\rho\sigma}\eta^{\rho\sigma} = \text{"}4\text{"}$)

Then
$$(**) = \frac{-2(k')^2}{4}\left[-2\not{\varepsilon}\right] - 2\left\{xy\,\not{p}'\not{\varepsilon}\not{p} - x(1-x)\not{p}'\not{\varepsilon}\not{p}' - y(1-y)\not{p}\not{\varepsilon}\not{p} + (1-x)(1-y)\not{p}\not{\varepsilon}\not{p}'\right\}$$

To evaluate the 2nd term, we exploit
$$\not{p}\,u(\vec{p}) = M\,u(\vec{p}) \quad \text{and} \quad \bar{u}(\vec{p}')\not{p}' = \bar{u}(\vec{p}')M.$$

So, when sandwiched between $\bar{u}(\vec{p}')$ and $u(\vec{p})$

$$(**) = (k')^2\not{\varepsilon} - 2M^2\not{\varepsilon}\left\{xy + x(1-x) + y(1-y) - (1-x)(1-y)\right\} + 2\cdot\overset{2M^2-q^2}{\overbrace{(2p\cdot p')}}(1-x)(1-y)\not{\varepsilon}$$
$$-2\left\{-x(1-x)M\,2(\varepsilon\cdot p') - y(1-y)\,2(\varepsilon\cdot p)M + (1-x)(1-y)\left(2(\varepsilon\cdot p) + 2(\varepsilon\cdot p')\right)M\right\}$$

So, the numerator becomes

$$\left[\bar{u}(\vec{p}')\not{\varepsilon}\,u(\vec{p})\right]\times\left(-2M^2\overset{\to\{-2+4(x+y)-(x+y)^2\}}{\left\{xy + x(1-x) + y(1-y) - 3(1-x)(1-y) + 1\right\}} - \left(k_E^2\right) - 2(1-x)(1-y)\,q^2\right)$$
$$+\left[\bar{u}(\vec{p}')\,u(\vec{p})\right]\times(-4M)\,\varepsilon\cdot(p+p')\left\{\underbrace{(1-x)(1-y) - \frac{x(1-x)+y(1-y)}{2}}_{=\frac{(1-x-y)(2-x-y)}{2}} - (1-x-y)\right\}$$
$$= (1-x-y)\frac{(x+y)}{2}$$

Let us parametrize

$$i\mathcal{M} =: i(-Qe)\,\bar{u}(\vec{p}')\left(\varepsilon_\mu \gamma^\mu F_1(q^2) - \frac{F_2(q^2)}{4M}\varepsilon_\mu[\gamma^\mu, \gamma^\nu]\, q_\nu\right) u(\vec{p})$$

$$= i(-Qe)\,\bar{u}(\vec{p}')\left(\varepsilon_\mu \gamma^\mu (F_1 + F_2) - \varepsilon_\mu (p'+p)^\mu \frac{F_2}{2M}\right) u(\vec{p})$$

using $\not{p}\, u(\vec{p}) = M u(\vec{p})$ and $\bar{u}(\vec{p}')\not{p}' = \bar{u}(\vec{p}') M$.

At tree-level, $F_1 = 1$ and $F_2 = 0$.

F_1 : Dirac's form factor
F_2 : Pauli's form factor

$\Big\}$ are non-trivial functions of q^2 at the quantum level.

(μ^I : regarded "elementary" in QED though)

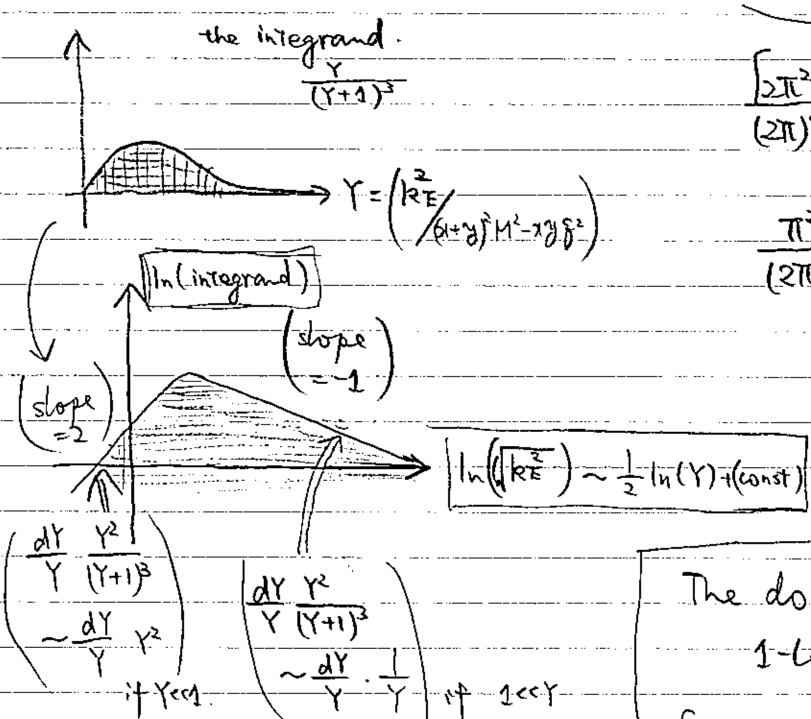
$$F_2^{(1\text{-loop})} = -2M \int \frac{d^4 k_E}{(2\pi)^4} \int_\Delta dx\,dy\, \frac{2(Qe)^2\; 2M\; (1-x-y)(x+y)}{-\left[(k_E^2) + (x+y)^2 M^2 + xy(-q^2)\right]^3}$$

Now we carry out d^4k_E integral

$$F_2^{(1)} = \int_{\Delta} dx\, dy \frac{(2M)^2(Qe)^2}{[(x+y)^2M^2 + xy(-q^2)]^3}\left\{\frac{2(1-x-y)}{(x+y)}\right\} \underbrace{\int \frac{d^4k_E}{(2\pi)^4}\frac{1}{\left[k_E^2/\left((x+y)^2M^2 - xyq^2\right) + 1\right]^3}}$$

$$= \frac{\boxed{2\pi^2 = \mathrm{vol}(S^3)}}{(2\pi)^4}\,\frac{1}{2}\int_0^{+\infty} d(k_E^2)\frac{(k_E^2)}{\left[\left(\frac{k_E^2}{(x+y)^2M^2 - xyq^2}\right)+1\right]^3}$$

$$= \frac{\pi^2}{(2\pi)^4}\left[(x+y)^2M^2 - xyq^2\right]^2 \cdot \underbrace{\int_0^{+\infty} dY \frac{Y}{[Y+1]^3}}_{=\,1/2}$$



> The dominant contribution to the 1-loop integral for $F_2^{(1)}$ is from $k_E \sim \mathcal{O}(M^2)$ or $\mathcal{O}(-q^2)$.

So $$F_2^{(1)} = \int_{\Delta} dx\, dy \frac{(Qe)^2}{(4\pi)^2}\frac{\frac{1}{2}(2M)^2\{2(1-x-y)(x+y)\}}{[(x+y)^2M^2 - xyq^2]}$$

$(x,y) \to (x,z)$, $z = (x+y)$

$$F_2^{(1)}(q^2=0) = \int_0^1 dz \frac{(Qe)^2}{(4\pi)^2}\frac{(2M)^2}{z^2M^2}(1-z)\, z \times \left(z \leftarrow \int_0^z dx\right)$$

$$= \frac{\alpha_e}{\pi} \times \frac{1}{2}.$$

The anomalous magnetic moment:

$$\boxed{g = 2 + 2F_2(q^2=0) \simeq 2 + \frac{\alpha_e}{\pi}} \quad \text{at 1-loop order.}$$

$\alpha_e \simeq 1/137$

It is also possible to compute $\left.\frac{\partial F_2}{\partial q^2}\right|_{q^2=0}$ and higher order

$\approx$ quantum "radius^2" of μ.