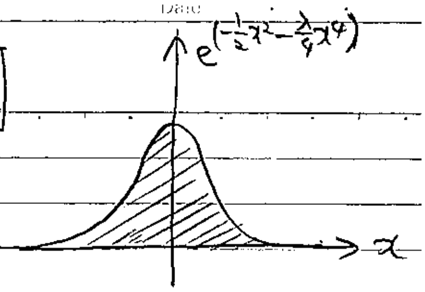
§ 9 A bonus track : Borel resummation



Consider an integral

$$Z(\lambda) := \int_{-\infty}^{+\infty} dx \; e^{\left(-\frac{1}{2}x^2-\frac{\lambda}{4}x^4\right)}.$$

This is like a path integral of ϕ^4-theory on $(-1)+1$-dim. space-time.

We know that $Z(\lambda=0)=\sqrt{2\pi}$, and it is a monotonically decreacing function of $\lambda \in \mathbb{R}_{\geq 0}$.

What if we Taylor-expand $e^{-\frac{\lambda}{4}x^4} = 1-\frac{\lambda}{4}x^4+\frac{1}{2}\left(\frac{\lambda}{4}\right)^2 x^8 - \cdots$

and evaluate the integral one by one? (as in perturbation theory)

$$Z[[\lambda]] := \sum_{n=0}^{\infty} \frac{(-\lambda/4)^n}{n!} \int_{-\infty}^{+\infty} dx \; e^{-\frac{1}{2}x^2} x^{4n}$$

$$= \sum_{n=0}^{\infty} \frac{(-\lambda/4)^n}{n!} \left[\int_0^{+\infty} dX \; e^{-X/2} X^{2n-\frac{1}{2}} = 2^{2n+\frac{1}{2}} \Gamma\left(2n+\frac{1}{2}\right) \right]$$

$$= \sum_{n=0}^{\infty} \frac{(-\lambda/4)^n}{n!} 2^{2n+\frac{1}{2}} \left(2n-\frac{1}{2}\right)\left(2n-\frac{3}{2}\right)\left(2n-\frac{5}{2}\right)\left(2n-\frac{7}{2}\right) \cdots \left(\frac{3}{2}\right)\left(\frac{1}{2}\right) \times \left[\Gamma(1/2)=\sqrt{\pi}\right]$$

$$= \sqrt{2\pi} \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} \left(\tfrac{1}{4}\right)_n \left(\tfrac{3}{4}\right)_n .$$

$$(\alpha)_n := \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)$$

The convergence radius of this power series is

$$|\lambda| < \lim_{n\to\infty} \left(\frac{(n+1)!}{\left(\frac{1}{4}\right)_{n+1}\left(\frac{3}{4}\right)_{n+1}} \cdot \frac{\left(\frac{1}{4}\right)_n\left(\frac{3}{4}\right)_n}{n!} \right) = \lim_{n\to\infty} \left(\frac{(n+1)}{\left(\frac{1}{4}+n\right)\left(\frac{3}{4}+n\right)} \right) = 0$$

So, for $0<\lambda\in\mathbb{R}$, $\boxed{\text{we cannot use the perturbative expansion } Z[[\lambda]] \text{ to obtain } Z(\lambda) \text{ with arbitrary high precision.}}$

Are there any ways we can still use the perturbative calculations ??

Here is an idea.

For a series $f[[\lambda]] = \sum_{n=0}^{\infty} c_n \lambda^n$, introduce $(\mathcal{B}f)(s) := \sum_{n=0}^{\infty} \frac{c_n}{n!} s^n$.

The new function $\mathcal{B}f$ is called the Borel transform of f.

☆ $\mathcal{B}f$ has a better chance of having non-zero convergence radius than the original f.

☆ The Laplace transform of $\mathcal{B}f$ is, if evaluated naively,

$$\int_0^{+\infty} ds\, e^{-s\left(\frac{1}{\lambda}\right)} \mathcal{B}f(s) \Rightarrow \sum_{n=0}^{\infty} \frac{c_n}{n!} \int_0^{+\infty} ds\, e^{-s/\lambda} s^n = \sum_{n=0}^{\infty} \frac{c_n}{n!} \lambda^{n+1} \cdot n!$$

$$= \lambda f[[\lambda]].$$

So, why don't we make sense of $Z[[\lambda]]$ by thinking of it as $\frac{1}{\lambda} \int_0^{+\infty} ds\, e^{-s(1/\lambda)} \mathcal{B}Z(s)$?

⟨ In QFT, we have to carry out infinite-dimensional integral (path integral) to determine $Z[[\lambda]]$. It is not a terrible set-back to bring in one-dim. integral. ⟩

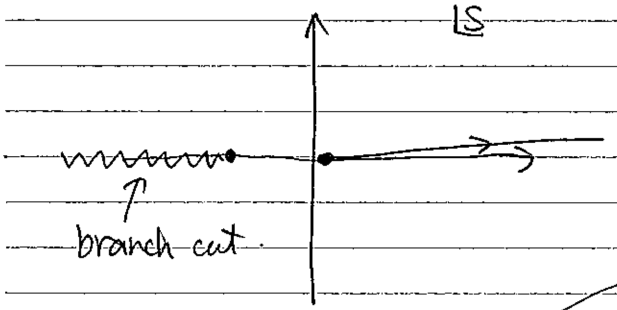
The idea on the example:

$$Z[[\lambda]] = \sqrt{2\pi} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{n!} (-\lambda)^n \Longrightarrow \mathcal{B}Z(s) = \sqrt{2\pi} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{n!\, n!} (-s)^n$$

$$\left(\begin{array}{c}\text{converges} \\ @\ |s|<1\end{array}\right) \longrightarrow \boxed{= \sqrt{2\pi}\, {}_2F_1\left(\tfrac{3}{4}, \tfrac{1}{4}; 1; -s\right)}$$

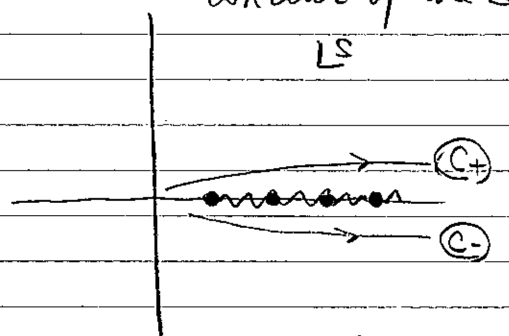
$$\boxed{Z(\lambda) \overset{?}{=} \frac{1}{\lambda} \int_0^{+\infty} ds\, e^{-s/\lambda} \sqrt{2\pi}\, {}_2F_1\left(\tfrac{3}{4}, \tfrac{1}{4}; 1; -s\right) =: \widehat{Z}(\lambda)}$$

☆ In the Borel plane (complex s-plane)



If $Bf(s)$ does not have a pole / cut along the real positive axis,
we say that $\begin{cases} f[[\lambda]] \text{ is Borel-summable} \\ \frac{1}{\lambda}\int_0^{+\infty} ds\, e^{-s/\lambda}\, Bf(s) \text{ as its Borel resummation.} \end{cases}$

☆ Even when there is a singularity in the real positive direction, we can define Borel resummations by choosing the integration contour of the Laplace transformation so the singularity is avoided.



$$\begin{cases} \hat{f}_{C_+}(\lambda) := \frac{1}{\lambda}\int_{C_+} ds\, e^{-s/\lambda}\, Bf(s) \\ \hat{f}_{C_-}(\lambda) := \frac{1}{\lambda}\int_{C_-} ds\, e^{-s/\lambda}\, Bf(s) \end{cases}$$

☆ Evaluate $\hat{f}_c(\lambda)$:

$$\lambda \hat{f}_c(\lambda) = \int_{C;\,|s|>(\text{radius}=s_*)(1-\delta)} ds\, e^{-s/\lambda}\, Bf(s) + \int_{C;\,|s|<s_*(1-\delta)} ds\, e^{-s/\lambda}\left(\sum_{n=0}^{s_*/\lambda}\frac{c_n}{n!}s^n\right) + \int_{C;\,|s|<s_*(1-\delta)} ds\, e^{-s/\lambda}\left(\sum_{n=s_*/\lambda}^{\infty}\frac{c_n}{n!}s^n\right)$$

e.g. when $\boxed{Bf(s) = \sqrt{2\pi}\,{}_2F_1\left(\frac{3}{4},\frac{1}{4},1;-s\right)}$

$\Downarrow$

$Bf(s) \sim s^{-1/4} \searrow 0$ along large real pos. axis.

$\Rightarrow \boxed{\approx e^{-(s_*/\lambda)}}$

evaluate each term.

$\boxed{\sum_{n=0}^{s_*/\lambda} c_n \lambda^{n+1}} \boxed{+\mathcal{O}(e^{-s_*/\lambda})}$

perturbative series

The integrand $e^{-s/\lambda}\left(\frac{s}{s_*}\right)^n \sim e^{-\frac{s}{\lambda}+n\ln(s/s_*)}$

has a peak @ $-\frac{1}{\lambda}+\frac{n}{s}=0 \Rightarrow$ $(s\sim n\lambda)$

$\boxed{s_{peak} < s_* \text{ iff } n < \frac{s_*}{\lambda}}$

$\boxed{< (1-\delta)^{\frac{s_*}{\lambda}}}$

$\boxed{\hat{f}_c(\lambda) = \text{pert. series} + \mathcal{O}(e^{-s_*/\lambda})}$

$\boxed{(\hat{f}_{C_+} - \hat{f}_{C_-}) \sim \mathcal{O}(e^{-s_*/\lambda})}$

Examples · $\frac{\lambda}{4}\phi^4$ on 0-dim. $\Rightarrow$ cut along $s \in [-\infty, -1]$

· $\frac{\lambda}{24}\phi^4$ on 4-dim. $\Rightarrow$ cut along $s \in [-\infty, -16\pi^2]$
(pole @?)
(see Weinberg §20.7)

· QCD poles on real positive axis (Weinberg §20.7)
· QED poles (Lautrup Phys. Lett. B69 ('77) 109)

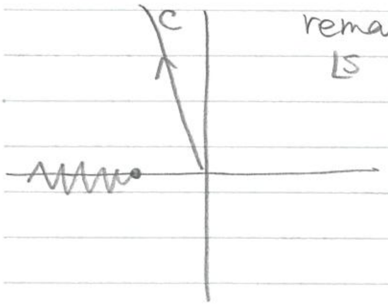
$\boxed{\text{There can be ambiguity in computing observables in a theory when singularity is on the real pos. axis in the Borel plane...}}$

| sounds ridiculous doesn't it? |

Back to the simplest example : $Z(\lambda) = \int_{-\infty}^{+\infty} dx \, e^{\left(-\frac{1}{2}x^2 - \frac{\lambda}{4}x^4\right)}$.

What if λ is not necessarily real positive, but $\lambda \in \mathbb{C}$?

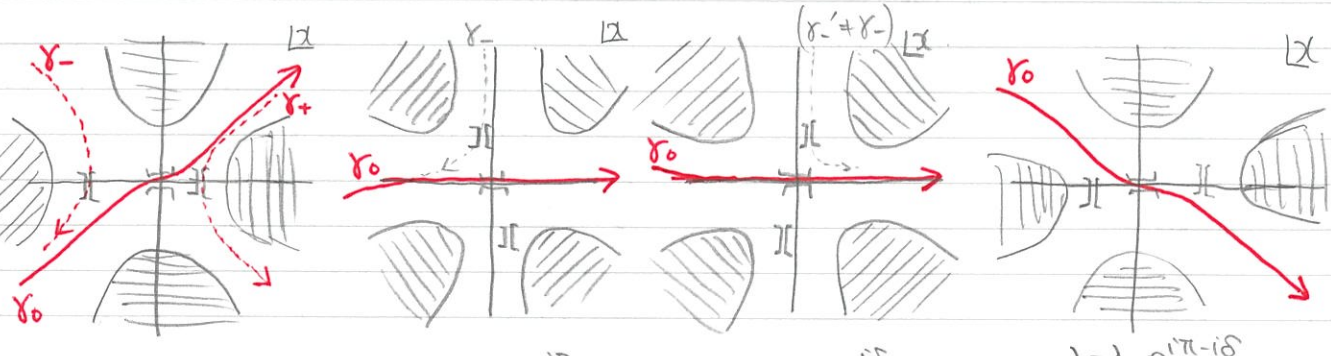
$$\hat{Z}(\lambda) := \frac{1}{\lambda}\int_C ds \, e^{-s/\lambda} BZ(s) \; ; \qquad BZ(s) = \sqrt{2\pi}\, {}_2F_1\!\left(\tfrac{3}{4}, \tfrac{1}{4}; 1; -s\right)$$



remains well-defined by choosing C in the direction of the phase of λ.

When λ is real negative, we have ambiguity in $\hat{Z}_C(\lambda)$
depending on whether we think of the phase of λ as $+\pi$ or $-\pi$.

Back in the path integral language, here is what is happening.



$\lambda = \lambda_0 e^{-i\pi + i\delta}$ $\qquad \lambda = \lambda_0 e^{-i\delta}$ $\qquad \lambda = \lambda_0 e^{i\delta}$ $\qquad \lambda = \lambda_0 e^{i\pi - i\delta}$

$$\boxed{Z_{\gamma_0}(\lambda = \lambda_0 e^{i\pi}) = Z_{\gamma_- + \gamma_0 + \gamma_+}(\lambda = \lambda_0 e^{-i\pi})}$$

There are stationary configurations in the action (saddles, I-$\bar{I}$) also called

$$-S = -\frac{1}{2}x^2 - \frac{\lambda}{4}x^4.$$

$$\frac{\partial S}{\partial x} = x + \lambda x^3 = 0 \Rightarrow \begin{cases} x_\pm = \pm i\sqrt{1/\lambda} \\ x = 0 \end{cases}$$

$$-\frac{\partial^2 S}{\partial x^2} = -(1+3\lambda x^2) = \begin{cases} 2 & \text{at } x = x_\pm \\ -1 & \text{at } x = 0 \end{cases} \Rightarrow \text{the steepest descent} \begin{cases} \updownarrow \\ \leftrightarrow \end{cases}$$

$$-S\Big|_{x_\pm} = -\frac{1}{2}\left(\frac{-1}{\lambda}\right) - \frac{\lambda}{4}\left(\frac{-1}{\lambda}\right)^2 = \frac{1}{4\lambda}$$

The ambiguity is $Z_{\gamma_0}(\lambda=\lambda_0 e^{i\pi}) - Z_{\gamma_0}(\lambda=\lambda_0 e^{-i\pi}) = Z_{\gamma_+ + \gamma_-}(\lambda=-\lambda_0) \sim e^{(1/4\lambda)} \ll 1$

From this example, we learn that the presence of saddles in path integration can be a cause of singularities in the Borel plane and the ambiguity in the Borel resummation.

How should this ambiguity be fixed?

This example does not help. (for $\lambda > 0$; there's no ambiguity)

just $Z_{\gamma_0}(\lambda = \lambda_0)$

Values of the coupling constant

- where the cycle γ of steepest descent reconnect
- where the Laplace transformation integration contour hits singularity in the Borel plane

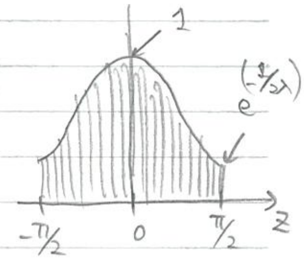
Ref: arXiv:1403.1277 (§2) by A. Cherman, D. Dorigoni, M. Ünsal

Another toy example

modified Bessel

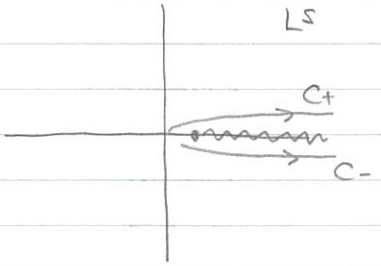
$$Z(\lambda) := \frac{1}{\sqrt{\lambda}} \int_{-\pi/2}^{\pi/2} dz \; e^{-\frac{1}{2\lambda}\sin^2(z)} = \frac{\pi e^{(-1/4\lambda)}}{\sqrt{\lambda}} I_0(1/4\lambda)$$

$$= \int_{-\pi/2\sqrt{\lambda}}^{+\pi/2\sqrt{\lambda}} dy \; e^{-\frac{1}{2\lambda}\sin^2(\sqrt{\lambda} y)} = \int_{-\pi/2\sqrt{\lambda}}^{\pi/2\sqrt{\lambda}} dy \; e^{\left(-\frac{1}{2}y^2 + \frac{\lambda}{6}y^4 - \frac{\lambda^2}{90}y^6 + \cdots\right)} .$$

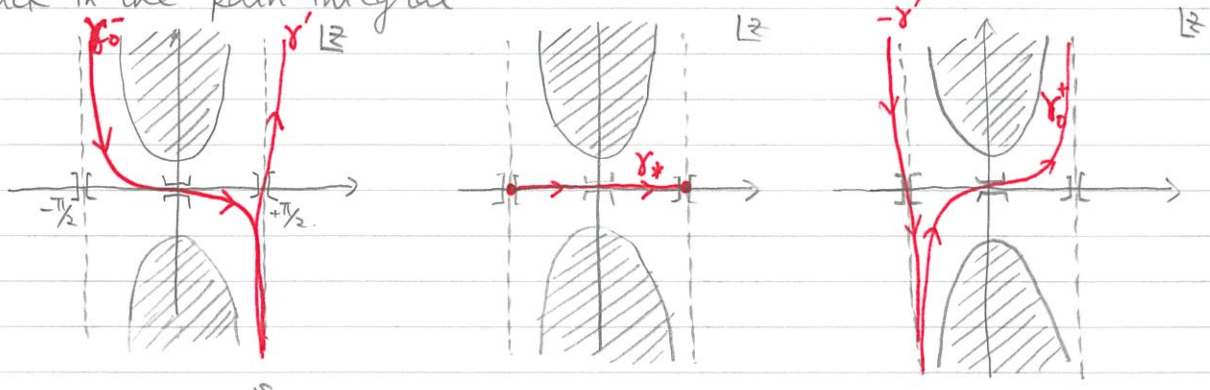


There should be no ambiguity in $Z(\lambda)$.

$$\begin{cases} Z[[\lambda]] : \text{zero convergence radius} \\ BZ(s) = \sqrt{2\pi}\; {}_2F_1\left(\frac{1}{2}, \frac{1}{2}, 1; 2s\right) \Rightarrow \text{branch cut along } s \in [1/2, +\infty]. \end{cases}$$



Back in the path integral



$\lambda = \lambda_0 e^{-i\delta}$ $\qquad$ $\lambda = \lambda_0$ $\qquad$ $\lambda = \lambda_0 e^{i\delta}$

$\gamma_* = \gamma_0^- + \gamma'$ $\qquad\qquad$ $\gamma_* = \gamma_0^+ + (-\gamma')$

$\Rightarrow$ We also need a perturbative series around $z = \pm\pi/2$ (another saddle).

$$\Rightarrow e^{-\frac{1}{2\lambda}} Z'[[\lambda]] \Rightarrow e^{-\frac{1}{2\lambda}} BZ'(s); \qquad BZ'(s) = \sqrt{2\pi}\; {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; -2s\right) \; (\times i?)$$

(no singularity along the real positive axis of s)

$$\boxed{\begin{aligned} \hat{Z}(\lambda) &:= \frac{1}{\lambda}\int_{C-} ds\, e^{-s/\lambda} BZ(s) + \frac{1}{\lambda} e^{-\frac{1}{2\lambda}} \int_0^{+\infty} ds\, e^{-s/\lambda} BZ'(s) = Z_{\gamma_0^-}(\lambda) + Z_{\gamma'}(\lambda) \\ &= \frac{1}{\lambda}\int_{C+} ds\, e^{-s/\lambda} BZ(s) - \frac{1}{\lambda} e^{-\frac{1}{2\lambda}} \int_0^{+\infty} ds\, e^{-s/\lambda} BZ'(s) = Z_{\gamma_0^+}(\lambda) - Z_{\gamma'}(\lambda) \end{aligned}}$$

There is no ambiguity, when this term is included.

$\boxed{\text{Summary}}$

Different cases

- Just one perturbative series is involved, and that is
  Borel-summable for physical values of couplings.
  (eg. $-S = -\frac{1}{2}x^2 - \frac{\lambda}{4}x^4$; $\lambda \geq 0$, or $\lambda\phi^4$-theory in 4-dim)
  [There may be other saddles
  There may be singularity in the Borel plane
  but they are not involved for physical coupling.]

- There are other saddles in the space (cycle) of path integral, and
  hence other perturbative series are associated with them.
  The primary power series is not Borel-summable for a physical coupling.
  (i.e. there remains ambiguity)
  For a coupling that is slightly different from the physical coupling by complex phase, we have an un-ambiguous combination of the primary and sub-leading power series, all of which are Borel-summable.
  Choice of the complex phase do not change the result.
  (eg. $-S = \frac{-1}{2\lambda}\sin^2(z)$ $\lambda \geq 0$) (expected: also QCD)

- QED
  * Its primary power series is known not to be Borel-summable.
    (a pole in the real pos. axis) (for $\alpha_e > 0$)
  * QED is expected not to be well-defined at $E > m_e e^{(3\pi/2\alpha_e)} =: \Lambda_{QED}$
    $\Rightarrow \left(\frac{p}{\Lambda_{QED}}\right)^{n \in \mathbb{N}}$ contributions to an observable
    look like $\left(e^{-\frac{3\pi}{2\alpha_e}\cdot n}\right)$ contributions.
    ⟨higher-dim. local operators added to QED⟩
    rather than saddles