convention:

$\eta^{\mu\nu} = \mathrm{diag}(+,-,-,-)$

$e > 0, \quad Q_e = -1.$

§0. Feynman rule.

When a Lagrangian is given in terms of fields and quantized, how do we compute correlation functions?

Example: $\phi(x)$: a complex scalar field

- Higgs doublet in the SM.
- charged pion $\pi^\pm$
- atoms (spin-0)
- ⋮

$$\mathcal{L} = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - V(\phi)$$

$\Longrightarrow$ use the bilinear part to quantize

$$\phi_I(x^\mu) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(\underline{\underline{e}}^{-ip\cdot x} a_{\vec{p}} + \underline{\underline{e}}^{ip\cdot x} b^\dagger_{\vec{p}} \right)$$

$$\phi^\dagger_I(x^\mu) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(\underline{\underline{e}}^{ip\cdot x} a^\dagger_{\vec{p}} + \underline{\underline{e}}^{-ip\cdot x} b_{\vec{p}} \right)$$

$$[a_{\vec{p}}, a^\dagger_{\vec{q}}] = (2\pi)^3 \delta^3(\vec{p} - \vec{q}) \qquad \underline{\underline{e}}^{\pm ip\cdot x} = e^{\pm ip\cdot x}\Big|_{p^0 = \sqrt{\vec{p}^2 + M^2}}$$

$$[b_{\vec{p}}, b^\dagger_{\vec{q}}] = (2\pi)^3 \delta^3(\vec{p} - \vec{q})$$

$\phi_I(x)$: interaction-picture operators

$$\phi_I(t, \vec{x}) := e^{iH_0(t-t_0)} \phi_I(t_0, \vec{x})\, e^{-iH_0(t-t_0)}$$

time-evolution under the bilinear Hamiltonian H_0.

Hope to be able to compute time-ordered correlation fcn's:

$$\langle 0 | T\left\{ \phi_I(x_1)\, \phi_I(x_2) \cdots \phi^\dagger_I(y_1)\, \phi^\dagger_I(y_2) \cdots \exp\left[-i\int d^4z\, V(\phi_I\text{'s})\right] \right\} | 0 \rangle$$

Scattering amplitudes, cross sections, decay rates etc. can be extracted from time-ordered correlation fcn's.

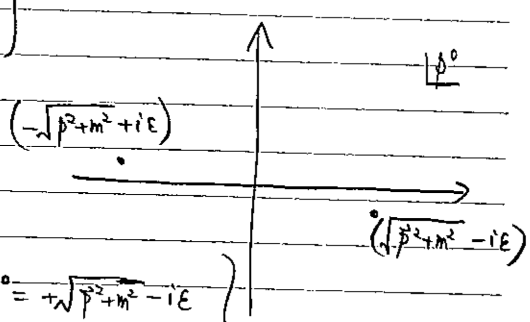
How to extract? $\Longrightarrow$ see Peskin–Schroeder §4.5 + §4.6.

Propagators : the easiest time-ordered correlation fcn's.

$$\langle 0|T\{\phi_I(x)\,\phi_I^\dagger(y)\}|0\rangle = \left\{\begin{array}{ll} \text{if } x^0>y^0 & \text{use } \langle 0|\,a^{\phi}_{\vec p}\;a^{\phi^\dagger\,\dagger}_{\vec q}|0\rangle \\ \text{if } y^0>x^0 & \text{use } \langle 0|\,b^{\phi^\dagger}_{\vec q}\;b^{\dagger\,\phi}_{\vec p}|0\rangle \end{array}\right\}$$

$$= \int \frac{d^3\vec p}{(2\pi)^3}\frac{1}{(2E_{\vec p})} \times \left\{\begin{array}{ll} \text{if } x^0>y^0 & e^{-ip\cdot(x-y)} \\ \text{if } y^0>x^0 & e^{ip\cdot(x-y)} \end{array}\right\}$$

$$= \int \frac{d^4p}{(2\pi)^4}\frac{i\,e^{-ip\cdot(x-y)}}{(p^2-m^2+i\epsilon)}$$



$$\left\{\begin{array}{l} \text{if } x^0>y^0 : \text{ pick up the Res @ } p^0 = +\sqrt{\vec p^2+m^2}-i\epsilon \\ \text{if } y^0>x^0 : \text{ pick up the Res @ } p^0 = -\sqrt{\vec p^2+m^2}+i\epsilon \end{array}\right\}$$

Its Fourier transform is

$$\boxed{\begin{array}{l} \int d^4x \int d^4y\; e^{ip\cdot x}\, e^{-iq\cdot y}\, \langle 0|T\{\phi_I(x)\,\phi_I(y)^\dagger\}|0\rangle \\ \quad = (2\pi)^4\delta^4(p-q)\,\dfrac{i}{(p^2-m^2+i\epsilon)} \end{array}}$$

The propagator of a Dirac fermion

$$\boxed{\begin{array}{l} \int d^4x \int d^4y\; e^{ip\cdot x}\, e^{-iq\cdot y}\, \langle 0|T\{\psi_I(x)\,\bar\psi_I(y)\}|0\rangle \\ \quad = (2\pi)^4\delta^4(p-q)\,\dfrac{i\,(p\cdot\gamma+m)}{(p^2-m^2+i\epsilon)} \end{array}}$$

$\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}_{4\times4}$

The propagator of massless vector fields

$$\boxed{\begin{array}{l} \int d^4x \int d^4y\; e^{ip\cdot x}\, e^{-iq\cdot y}\, \langle 0|T\{A^a_{\mu,I}(x)\,A^b_{\nu,I}(y)\}|0\rangle \\ \quad = (2\pi)^4\delta^4(p-q)\,\dfrac{-i\eta_{\mu\nu}\,\delta^{ab}}{(p^2+i\epsilon)} \qquad \text{(Feynman gauge)} \end{array}}$$

((See Peskin-Schroeder §2 (scalar) §3 (Dirac fermion) §9 (massless vector)
§21 (massive vector)))

Example : QED

$$\mathcal{L} = \bar{\Psi}\left[i\gamma^\mu(\partial_\mu + ieQA_\mu) - m\right]\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}.$$

Dirac (4-component) fermion for $e^\pm$, or for $\mu^\pm$

photon. $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

Consider

$$\langle 0| T\left\{ \Psi_I(x_3)\Psi_I(x_4)\bar{\Psi}_I(x_1)\bar{\Psi}_I(x_2) \exp\left[-ieQ\int d^4z\,(\bar{\Psi}_I\gamma^\mu\Psi_I)A_{\mu,I}\right]\right\}|0\rangle$$
$$\int d^4x_3\int d^4x_4\int d^4x_1\int d^4x_2\; e^{ip_3\cdot x_3}e^{ip_4\cdot x_4}e^{-ip_1\cdot x_1}e^{-ip_2\cdot x_2} \qquad (*)$$

↳ a function of $(p_1^\mu, p_2^\mu, p_3^\mu, p_4^\mu)$.

Expand the exp[----] into $1 + \left[-ieQ\int d^4z\,(\bar{\Psi}\gamma^\mu\Psi A_\mu)(z)\right] + \frac{1}{2}\left[(-ieQ)^2\int d^4z_1(\bar{\Psi}\gamma^\mu\Psi A_\mu)(z_1)\int d^4z_2(\bar{\Psi}\gamma^\nu\Psi A_\nu)(z_2)\right]$

$+ \cdots\cdots$

$$\left(\begin{aligned}&(2\pi)^4\delta^4(p_4-p_1)(2\pi)^4\delta^4(p_3-p_2)\frac{i[p_1\cdot\gamma+m]}{(p_1^2-m^2+i\epsilon)}\frac{i[p_2\cdot\gamma+m]}{(p_2^2-m^2+i\epsilon)}\\ &\ominus(2\pi)^4\delta^4(p_4-p_2)(2\pi)^4\delta^4(p_3-p_1)\frac{i[p_1\cdot\gamma+m]}{(p_1^2-m^2+i\epsilon)}\frac{i[p_2\cdot\gamma+m]}{(p_2^2-m^2+i\epsilon)}\end{aligned}\right)$$

Ψ's & $\bar{\Psi}$'s anti-commute.

⟨not interesting though.⟩

Note: only the terms of the form
$\langle 0|\; a^n a^{\dagger n}\; b^m b^{\dagger m}\; c^k c^{\dagger k}|0\rangle$ contribute.

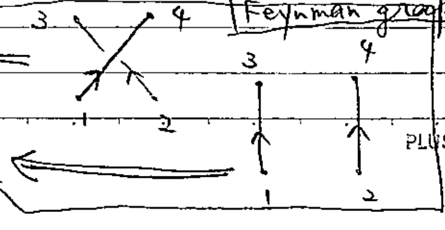
$$a_3a_4a_1^\dagger a_2^\dagger = a_3\{a_4, a_1^\dagger\}a_2^\dagger - a_3a_1^\dagger a_4 a_2^\dagger$$
$$= a_3a_2^\dagger\{a_4, a_1^\dagger\} - a_3a_1^\dagger\{a_4, a_2^\dagger\} + a_3a_1^\dagger a_2^\dagger a_4$$
$$= \{a_3, a_2^\dagger\}\{a_4, a_1^\dagger\} - a_2^\dagger a_3\{a_4, a_1^\dagger\} - \{a_3, a_1^\dagger\}\{a_4, a_2^\dagger\}$$
$$+ a_1^\dagger a_3\{a_4, a_2^\dagger\} + a_3a_1^\dagger a_2^\dagger a_4$$

contraction pattern.

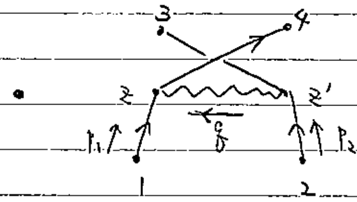
$a_3a_4a_1^\dagger a_2^\dagger \Rightarrow \{a_3, a_2^\dagger\}\{a_4, a_1^\dagger\}$

$a_3a_4a_1^\dagger a_2^\dagger \Rightarrow -\{a_3, a_1^\dagger\}\{a_4, a_2^\dagger\}$

Feynman graph



The first interesting contributions to (*) are



$$(-ieQ)^2\left(\frac{i[p_4\cdot\gamma+m]}{(p_4^2-m^2+i\epsilon)}\gamma^\mu\frac{i[p_1\cdot\gamma+m]}{(p_1^2-m^2+i\epsilon)}\right)\left(\frac{i[p_3\cdot\gamma+m]}{(p_3^2-m^2+i\epsilon)}\gamma^\nu\frac{i[p_2\cdot\gamma+m]}{(p_2^2-m^2+i\epsilon)}\right)$$

$$\int dz\, e^{ip_4\cdot z}e^{-ip_1\cdot z}\int dz'\, e^{ip_3\cdot z'}e^{-ip_2\cdot z'}\int\frac{d^4q}{(2\pi)^4}\frac{-i\eta_{\mu\nu}\,e^{-iq\cdot(z-z')}}{(q^2+i\epsilon)}$$

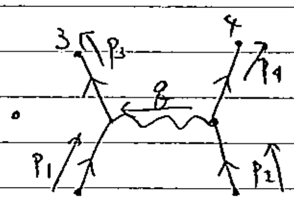
$$= (2\pi)^4\delta^4(p_4+p_3-p_1-p_2)\times$$

$$\left(\frac{i[\not{p}_4+m]}{(p_4^2-m^2+i\epsilon)}[-ieQ\gamma^\mu]\frac{i[\not{p}_1+m]}{(p_1^2-m^2+i\epsilon)}\right)\left(\frac{i[\not{p}_3+m]}{(p_3^2-m^2+i\epsilon)}[-ieQ\gamma^\nu]\frac{i[\not{p}_2+m]}{(p_2^2-m^2+i\epsilon)}\right)\frac{-i\eta_{\mu\nu}}{(q^2+i\epsilon)}\Bigg|_{q^\mu=(p_2-p_3)^\mu=(p_4-p_1)^\mu}$$

(notation $\not{p} := p_\mu\gamma^\mu$)

Π(propagators) $\cdot$ Π(vertex rules) $\cdot$ (overall momentum conservation)

momentum conservation is $\left\{\begin{matrix}\text{satisfied}\\ \text{enforced}\end{matrix}\right\}$ at each interaction vertex.



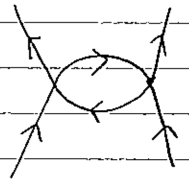
$$-\left(\frac{i[\not{p}_3+m]}{(p_3^2-m^2+i\epsilon)}(-ieQ\gamma^\mu)\frac{i[\not{p}_1+m]}{(p_1^2-m^2+i\epsilon)}\right)\left(\frac{i[\not{p}_4+m]}{(p_4^2-m^2+i\epsilon)}(-ieQ\gamma^\nu)\frac{i[\not{p}_2+m]}{(p_2^2-m^2+i\epsilon)}\right)\frac{-i\eta_{\mu\nu}}{(p_2-p_4)^2+i\epsilon}$$

The two terms above are added to get the $\mathcal{O}(e^2)$ contribution to the time-ordered correlation functions.

What happens if we want to compute contributions to time-ordered correlation fcn's like

in QED

or



in scalar $V(\phi) = \frac{\lambda}{4}(\phi)^4$ theory?

Suppose that momenta are specified (not the space-time coordinates) for all the external lines.

The # of freely chosen momenta

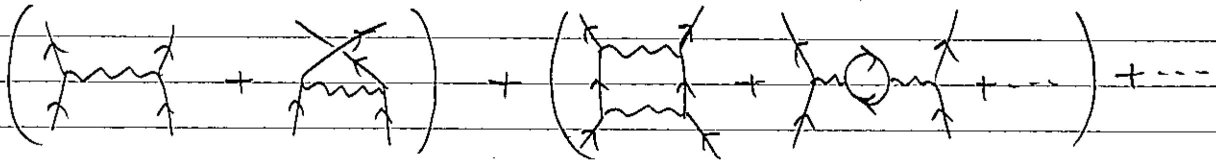
= #(of propagators) − #(of interaction vertices)

single field has all the possible momenta. — momentum conservation

− #(of connected components of the graph)

topology #(of loops in the graph)

So .... to a given time-ordered correlation fcn's (such as (*))



$\mathcal{O}(e^2)$ $\mathcal{O}(e^4)$ should add up

Computation of those contributions involves

$$\int \frac{d^4 q}{(2\pi)^4} \left(\text{propagators, vertex factors}\right)_{(q^\mu)}$$

loop momentum integration