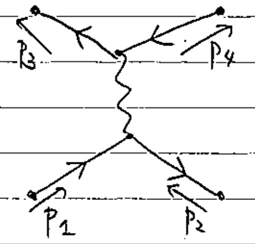
§4. Ultraviolet Divergence and Regularization

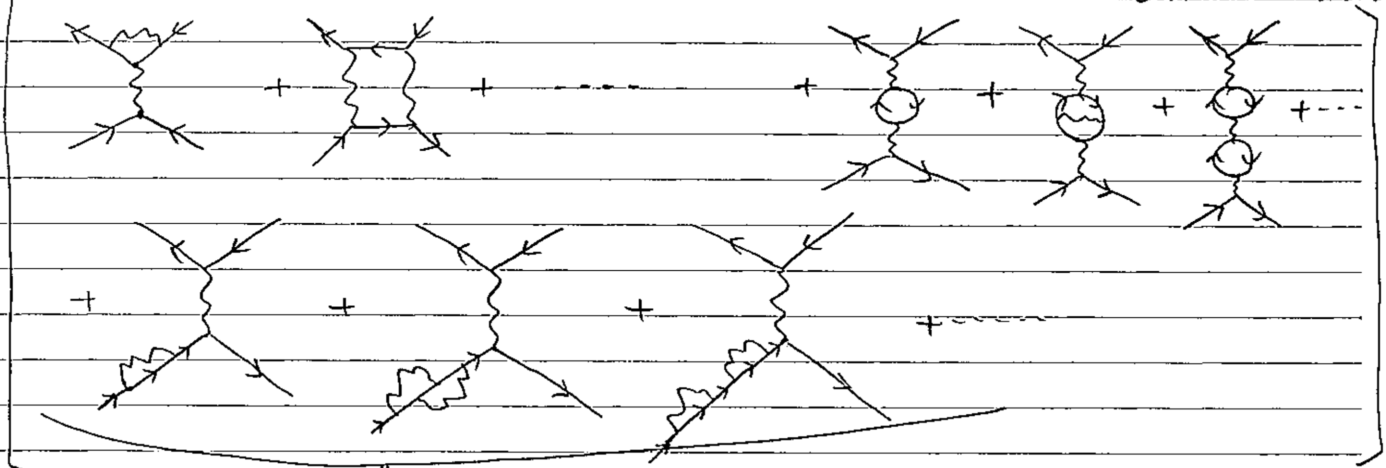
§4.1. An example : Self-Energy

Consider the $e^- + e^+ \to \mu^- + \mu^+$ process in QED.



$$= \frac{i[\not p_3 + M]}{(p_3^2 - M^2 + i\varepsilon)}(-ieQ_{(\mu)}\gamma^{\kappa})\frac{i[\not p_4 + M]}{(p_4^2 - M^2 + i\varepsilon)}$$
$$\frac{i[\not p_2 + m]}{(p_2^2 - m^2 + i\varepsilon)}(-ieQ_{(e)}\gamma^{\lambda})\frac{i[\not p_1 + m]}{(p_1^2 - m^2 + i\varepsilon)}$$
$$\times \frac{(-i\eta_{\kappa\lambda})}{(p_1 + p_2)^2 + i\varepsilon}(2\pi)^4\delta^4(p_3 + p_4 - p_1 - p_2).$$

This is the leading order ($\mathcal{O}(e^2)$) contribution to the 4-point time-ordered correlation function, but there are other contributions.



An observation : those contributions have the following structure

$$(\text{a common factor}) \times \Bigg[\underbrace{\frac{i(\not p_1 + m)}{(p_1^2 - m^2 + i\varepsilon)}}_{\text{leading order}} + \frac{i(\not p_1 + m)}{(p_1^2 - m^2 + i\varepsilon)} \times (\mathcal{M}_1) \times \frac{i(\not p_1 + m)}{(p_1^2 - m^2 + i\varepsilon)} + \frac{i(\not p_1 + m)}{(p_1^2 - m^2 + i\varepsilon)} \times (\mathcal{M}_1') \times \frac{i(\not p_1 + m)}{(p_1^2 - m^2 + i\varepsilon)}$$
$$+ \frac{i(\not p_1 + m)}{(p_1^2 - m^2 + i\varepsilon)} \times \mathcal{M}_1 \times \frac{i(\not p_1 + m)}{(p_1^2 - m^2 + i\varepsilon)} \times \mathcal{M}_1 \frac{i(\not p_1 + m)}{(p_1^2 - m^2 + i\varepsilon)}$$
$$+ \cdots$$

$$\cong (\text{the common factor}) \times \frac{i}{(\not p_1 - m + i\varepsilon) - i(\mathcal{M}_1 + \mathcal{M}_1')}$$

(after integrating loop momenta)

$\mathcal{M}_1 + \mathcal{M}_1'$: 4×4 matrix valued function of p_1^μ and m.

So, it is better to deal with perturbative corrections

— at the core of a particle-particle interaction

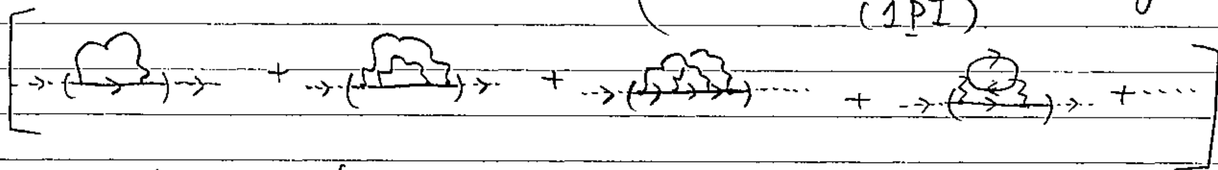
- the 1st unseparable corrections on an external state
- the 2nd unseparable :
- the 3rd unseparable :

⋮

→ can be organized into a geometric series.

⇒ just enough to compute single unseparable corrections.

(1 particle irreducible diagrams (1PI))



$= \sum_{\text{sum}} (\mathcal{M}\text{'s})$ (including the vertex factors at the both ends but {not including $(2\pi)^4 \delta^4(p_1^{in} - p_1^{out})$, not including $\frac{i(\not{p}_1 + m)}{(p_1^2 - m^2 + i\varepsilon)}$ at the both ends})

$$=: -i\left[\not{p}_1 A(p_1^2, m^2) + B(p_1^2, m^2)\right]$$

So, the 4-point time-ordered correlation function is in the form of

$$(\text{the common factor}) \times \left[\frac{i}{(1 - A(p_1^2, m^2))\not{p}_1 - (m+B) + i\varepsilon}\right].$$

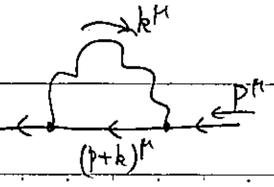
$$\parallel$$

$$\frac{i\left[(1-A)\not{p}_1 + (m+B)\right]}{\left[(1-A)^2 p_1^2 - (m+B)^2 + i\varepsilon\right]}$$

[A pole in $(p_1)^2$ ⟺ an on-shell particle in the Hilbert space

The residue ⟺ relative normalization between fields and states.]

- The particle mass (where the pole is located) is shifted from the one we started with because of the interaction.
- What we see as a particle is NOT an elementary particle but a quantum mechanical cloud of interacting fields presenting themselves as a pole in the momentum space.
- Don't we need consistency in the calculation ??



$(A_\kappa \bar\psi \gamma^\kappa \psi)(z_2)\ (\bar\psi \gamma^\lambda \psi A_\lambda)(z_1)$

$$-i\left[A(p,m)\,\not p + B(p,m)\right]^{1\text{-loop}} = \int \frac{d^4k}{(2\pi)^4} (-ieQ\gamma^\kappa)\frac{i\left[(\not p+\not k)+m\right]}{(p+k)^2-m^2+i\varepsilon}(-ieQ\gamma^\lambda)\frac{(-i\eta_{\kappa\lambda})}{(k^2+i\varepsilon)}$$

§4.2 Evaluate this 1-loop self-energy

$$\left[\not p A + B\right] = -i(eQ)^2\int\frac{d^4k}{(2\pi)^4}\frac{\left\{\gamma^\kappa\left[(\not p+\not k)+m\right]\gamma_\kappa = -2(\not p+\not k)+4m\right\}}{\left[(p+k)^2-m^2+i\varepsilon\right]\left[k^2+i\varepsilon\right]}$$

$$\left\{\begin{array}{l}\gamma^\kappa\gamma^\nu\gamma_\kappa = -2\gamma^\nu \\ \gamma^\mu\gamma_\mu = 4\,\mathbb{1}_{4\times4}\end{array}\right\}$$

Calculation trick ①

$$\frac{1}{\alpha\beta} = \int_0^1 dx\,\frac{1}{[x\alpha+(1-x)\beta]^2} \qquad \left(\text{RHS} = \int_0^1 dx\,\frac{1}{(\alpha-\beta)^2}\frac{1}{\left(x+\frac{\beta}{\alpha-\beta}\right)^2} = \frac{1}{(\alpha-\beta)^2}\left(\frac{\alpha-\beta}{\beta}-\frac{\alpha-\beta}{\alpha}\right) = (\text{LHS})\right)$$

$$\frac{1}{\alpha\beta\gamma} = \int dx\int dy\,\frac{2}{[x\alpha+y\beta+(1-x-y)\gamma]^3} \qquad \left(\text{proof: similar} \quad \text{QFT II. week ⑮}\right)$$

$$\frac{1}{\alpha_1\alpha_2\cdots\alpha_n} = \int_{0\le x_i} d^nx\,\frac{\delta(\Sigma_i x_i - 1)\,(n-1)!}{[x_1\alpha_1+x_2\alpha_2+\cdots+x_n\alpha_n]^n}$$

So

$$\left[\not p A + B\right] = -i(eQ)^2\int\frac{d^4k}{(2\pi)^4}\int_0^1 dx\,\frac{\left[-2(\not p+\not k)+4\right]}{\left[x(p+k)^2-xm^2+(1-x)k^2+i\varepsilon\right]^2}$$

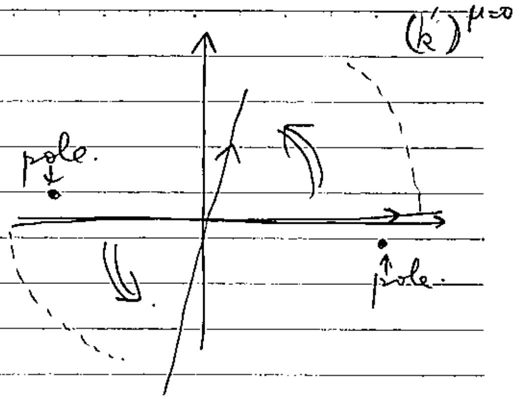
denominator: $k^2+2x\,k\cdot p + x p^2 - xm^2 = \underbrace{(k+xp)^2}_{(k')^\mu} + (x-x^2)p^2 - xm^2$

$$= -i(eQ)^2\int_0^1 dx\int\frac{d^4k'}{(2\pi)^4}\frac{\left[-2\not k' - 2(1-x)\not p + 4\right]}{\left[(k')^2+x(1-x)p^2-m^2+i\varepsilon\right]^2}$$

Because the denominator is even under $k'^\mu \to -k'^\mu$, just drop the $(-2\not k')$ term in the numerator.
(odd)

(x's introduced here: Feynman parameters)

Calculation trick ②



$$\int_{-\infty}^{+\infty}\frac{d(k')^0}{2\pi}\frac{1}{\left[[(k')^0]^2-|\vec{k}'|^2+(\text{something})+i\varepsilon\right]^{\nu}}$$

rotate the integration contour

(Wick rotation)

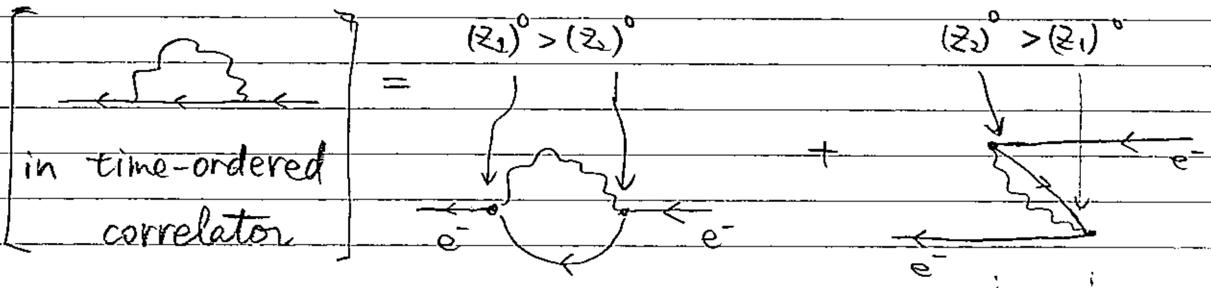
$$\int_{-\infty}^{+\infty}d(k')^0 \Rightarrow \int_{-i\infty}^{+i\infty}d(k')^0 = i\int_{-\infty}^{+\infty}dk^4 \qquad \text{Euclidean signature}$$

$$[\not{p}A+B]=\underbrace{[-i(eQ)^2](+i)}_{+(eQ)^2}\int_0^1 dx\int\frac{d^4k_E}{(2\pi)^4}\frac{[-2\not{p}(1-x)+4]}{[-k_E^2-xm^2+x(1-x)p^2]^2}$$

From the region $|k_E|^2 \gg m^2, |p^2|$, this integration diverges.

$$\boxed{e^2\int d|k_E|\left(k_E^3/(k_E)^4\right)\sim \underline{\text{log divergence}}}$$

Remember that



$(e^-+\gamma)$ state

off-shell

$e^-:(\vec{p}+\vec{k})\quad \gamma:(-\vec{k})$

$\sqrt{m^2+|\vec{p}+\vec{k}|^2}+|\vec{k}|$

$\neq p^0$ even when $(p^0)^2=|\vec{p}|^2+m^2$.

$\left[\begin{array}{cccc} e & +\ e^+ & +\ \gamma & +\ e \\ (\vec{p}) & (-\vec{p}-\vec{k}) & (\vec{k}) & (\vec{p}) \end{array}\right]$ intermediate state

off-shell

Ultraviolet

(Divergence) : there are just to many intermediate states for the correlator to be finite.

labeled by $\vec{k}\in\mathbb{R}^3$

§4.3 Regularization

UV divergence is from (many) high-energy states.

But we do not know for sure whether they are really "there".

- the Standard model : not sure beyond TeV
  certainly not above $M_{pl} \sim 2\times10^{18}$ GeV.
- QED : not above 80 GeV ($m_{W^\pm}$)
- $\pi^\pm$, N : not above 700 GeV ($\rho^\pm$, ρ^0)
- condensed matter system : not $|\vec{k}| > (1/a_B)$
  Brillouis zone boundary
  (lattice spacing
  not continuous)

$\boxed{\text{Do we need to know the right UV modification to those theories??}}$

No! $\boxed{\text{regularization + renormalization}}$

Modify QED Lagrangian at high-energy

eg. $$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4\Lambda^2}F_{\mu\nu}D^\kappa D_\kappa F^{\mu\nu} + \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi$$

- higher covariant derivative
- Pauli-Villars
- string theory .....

their combination ....

Now
$$[\not{p}A + B] \Rightarrow -i(eQ)^2 \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\kappa[(\not{p}+\not{k})+m]\gamma_\kappa}{\left[(p+k)^2 - m^2 + i\varepsilon\right]\left[k^2 - \frac{k^4}{\Lambda^2} + i\varepsilon\right]}$$

$\to (-k^2)(k^2-\Lambda^2)/\Lambda^2$

$$= -i(eQ)^2 \int \frac{d^4k}{(2\pi)^4} \int_\Delta dx\,dy \frac{[-2(\not{p}+\not{k}) + 4m]\quad(-2\Lambda^2)}{\left[k^2 + 2xk\cdot p - xm^2 - y\Lambda^2 + xp^2 + i\varepsilon\right]^3}$$

$$= -i(eQ)^2 \int_\Delta dx\,dy \int \frac{d^4k'}{(2\pi)^4} \frac{[-2(1-x)\not{p} + 4m](-2\Lambda^2)}{\left[(k')^2 - xm^2 - y\Lambda^2 + x(1-x)p^2 + i\varepsilon\right]^3}$$

PLUS

Wick rotation: then

$$\left[\not{p}A + B\right] = -i(eQ)^2 \int dx\, dy \; i \int \frac{d^4k_E}{(2\pi)^4} \frac{(-2\Lambda^2)}{(-)} \frac{\left[-2(1-x)\not{p} + 4m\right]}{\left[k_E^2 + xm^2 + y\Lambda^2 - x(1-x)p^2\right]^3}$$

$$= \frac{(eQ)^2}{(2\pi)^4}\left[2\pi^2 = \mathrm{vol}(S^3)\right] \frac{1}{2} \int_0^{+\infty} d(k_E^2)(k_E^2)\,(2\Lambda^2) \frac{\left[-2(1-x)\not{p} + m4\right]}{\left[(k_E^2) + xm^2 + y\Lambda^2 - x(1-x)p^2\right]^3}$$

$$= \frac{(eQ)^2}{16\pi^2} \int_0^{+\infty} dK \frac{K}{\left[K + xm^2 + y\Lambda^2 - x(1-x)p^2\right]^3} (2\Lambda^2)\left[-2(1-x)\not{p} + 4m\right]$$

$$\text{Now} \int_0^{+\infty} dK \frac{K}{\left[K + (xxx)\right]^3} = \frac{1}{(xxx)} \int_0^{+\infty} dk \frac{k+1-1}{(k+1)^3} = \frac{1}{(xxx)}\left(1 - \frac{1}{2}\right) = \frac{1}{2(xxx)}$$

converges

$$= \frac{\alpha Q^2}{4\pi} \int_0^1 dx \int_0^{(1-x)} dy \frac{\Lambda^2 \left[-2(1-x)\not{p} + 4m\right]}{\left[xm^2 + y\Lambda^2 - x(1-x)p^2\right]}$$

y-integral then

$$= \frac{\alpha Q^2}{4\pi} \int_0^1 dx \left[-2(1-x)\not{p} + 4m\right] \ln\left(\frac{(1-x)\Lambda^2 + xm^2 - x(1-x)p^2}{xm^2 - x(1-x)p^2}\right)$$

We found that

$$\boxed{\begin{aligned} A(p^2, m^2) &= \frac{\alpha Q^2}{4\pi} \int_0^1 dx \; -2(1-x) \ln\left(\frac{(1-x)\Lambda^2 + xm^2 - x(1-x)p^2}{xm^2 - x(1-x)p^2}\right) \\ B(p^2, m^2) &= \frac{\alpha Q^2}{4\pi} \int_0^1 dx \; 4m \ln\left(\frac{(1-x)\Lambda^2 + xm^2 - x(1-x)p^2}{xm^2 - x(1-x)p^2}\right) \end{aligned}}$$

A and B remain finite so long as Λ is finite (not infinite)

The UV divergence is under control.

(regularization)