§4.3 Regularization

UV divergence is from (many) high-energy states.

But we do not know for sure whether they are really "there".

- the Standard model : not sure beyond TeV
  certainly not above $M_{pl} \sim 2\times10^{18}$ GeV.
- QED : not above 80 GeV ($m_{W^\pm}$)
- $\pi^\pm$, N : not above 700 GeV ($\rho^\pm\ \rho^0$)
- condensed matter system : not $|\vec{k}| > (1/a_B)$
  Brillouis zone boundary
  (lattice spacing
  not continuous)

$\boxed{\text{Do we need to know the right UV modification to those theories??}}$

No! $\boxed{\text{regularization + renormalization}}$

Modify QED Lagrangian at high-energy

eg. $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4\Lambda^2}F_{\mu\nu}D^\kappa D_\kappa F^{\mu\nu} + \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi$

- higher covariant derivative
- Pauli-Villars
- string theory .....

their combination ....

Now

$$[\cancel{p}A + B] \Rightarrow -i(eQ)^2 \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\kappa[(\cancel{p}+\cancel{k})+m]\gamma_\kappa}{[(p+k)^2 - m^2 + i\varepsilon]\left[k^2 - \frac{k^4}{\Lambda^2} + i\varepsilon\right]} \qquad \to (-k^2)(-k^2-\Lambda^2)/\Lambda^2$$

$$= -i(eQ)^2 \int \frac{d^4k}{(2\pi)^4} \int_\Delta dx\,dy \frac{[-2(\cancel{p}+\cancel{k})+4m]\quad(-2\Lambda^2)}{[k^2+2xk\cdot p - xm^2 - y\Lambda^2 + xp^2 + i\varepsilon]^3}$$

$$= -i(eQ)^2 \int_\Delta dx\,dy \int \frac{d^4k'}{(2\pi)^4} \frac{[-2(1-x)\cancel{p}+4m](-2\Lambda^2)}{[(k')^2 - xm^2 - y\Lambda^2 + x(1-x)p^2 + i\varepsilon]^3}$$

PLUS

Wick rotation: then.

$$[\not{p}A+B] = -i(eQ)^2\int dx\,dy\; i\int\frac{d^4k_E}{(2\pi)^4}\frac{(-2\Lambda^2)}{(-)}\frac{[-2(1-x)\not{p}+4m]}{[k_E^2+xm^2+y\Lambda^2-x(1-x)p^2]^3}$$

$$= \frac{(eQ)^2}{(2\pi)^4}\left[2\pi^2 = vol(S^3)\right]\frac{1}{2}\int_0^{+\infty} d(k_E^2)(k_E^2)\,(2\Lambda^2)\frac{[-2(1-x)\not{p}+m4]}{[(k_E^2)+xm^2+y\Lambda^2-x(1-x)p^2]^3}$$

$$= \frac{(eQ)^2}{16\pi^2}\int_0^{+\infty}\frac{dK\,K}{[K+xm^2+y\Lambda^2-x(1-x)p^2]^3}(2\Lambda^2)[-2(1-x)\not{p}+4m]$$

Now $\displaystyle\int_0^{+\infty}\frac{dK\,K}{[K+(xxx)]^3} = \frac{1}{(xxx)}\int_0^{+\infty}dK\frac{K+1-1}{(K+1)^3} = \frac{1}{(xxx)}\left(1-\frac{1}{2}\right) = \frac{1}{2(xxx)}$ converges

$$= \frac{\alpha Q^2}{4\pi}\int_0^1 dx\int_0^{(1-x)}dy\,\frac{\Lambda^2\,[-2(1-x)\not{p}+4m]}{[xm^2+y\Lambda^2-x(1-x)p^2]}$$

y-integral then

$$= \frac{\alpha Q^2}{4\pi}\int_0^1 dx\,[-2(1-x)\not{p}+4m]\ln\left(\frac{(1-x)\Lambda^2+xm^2-x(1-x)p^2}{xm^2-x(1-x)p^2}\right)$$

We found that

$$\boxed{\begin{aligned} A(p^2,m^2) &= \frac{\alpha Q^2}{4\pi}\int_0^1 dx\; -2(1-x)\ln\left(\frac{(1-x)\Lambda^2+xm^2-x(1-x)p^2}{xm^2-x(1-x)p^2}\right)\\ B(p^2,m^2) &= \frac{\alpha Q^2}{4\pi}\int_0^1 dx\; 4m\;\ln\left(\frac{(1-x)\Lambda^2+xm^2-x(1-x)p^2}{xm^2-x(1-x)p^2}\right)\end{aligned}}$$

A and B remain finite so long as Λ is finite (not infinite)

The UV divergence is under control. (regularization)

Regularization methods may be different from the true mechanism adopted by the nature in rendering amplitudes UV finite. This is not a problem (as we will see later), because the choice of regularizatio[n] method does not matter after completing the procedure of renormalization.

§5 Renormalized Perturbation Theory

§5.1 Idea

We have seen that

$$\begin{cases} \bullet \displaystyle\int\frac{d^4p}{(2\pi)^4}\int\frac{d^4q}{(2\pi)^4}\, e^{ip\cdot x}\, e^{-iq\cdot y} \langle 0|T\{\Psi_I(x)\,\bar{\Psi}_I(y)\,\exp\left[i\int d^4z\,\mathcal{L}_{int,I}(z)\right]\}|0\rangle_{\text{connected}} \\ \quad = (2\pi)^4\delta^4(p-q)\,\dfrac{i}{[\not{p}-m-(A\not{p}+B)+i\varepsilon]} =: (2\pi)^4\delta^4(p-q)\left(\dfrac{i\not{p}\cdot Z_2}{p^2-m_{pole}^2+i\varepsilon}+\cdots\right) \\ \bullet\ \Psi_I(x) = \displaystyle\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left(u(\vec{p})_s\, \underline{\underline{e^{-ip\cdot x}}}\, \underset{(\text{free})}{a_{\vec{p},s}} + v(\vec{p})_r\, \underline{\underline{e^{ip\cdot x}}}\, \underset{(\text{free})}{b^\dagger_{\vec{p},s}}\right) \end{cases}$$

$\Rightarrow$ Renormalized fields $\dfrac{1}{\sqrt{Z_2}}\Psi_I(x) =: [\Psi_I]_r(x)$ and $\dfrac{1}{\sqrt{Z_2}}\bar{\Psi}_I = \overline{[\Psi_I]_r}$

have the right normalization.

also a theory parameter "m" should be chosen

so that the "m_{pole}" $\left(\begin{array}{l}(1-A(p^2,m^2))\,p^2-(m+B(p^2,m^2))^2=0 \\ \qquad\text{has a solution } p^2=m_{pole}^2\end{array}\right)$

has the right value.

QED action

$$\mathcal{L} = \bar{\Psi}\left[i\gamma^\mu(\partial_\mu+ieQA_\mu)-m\right]\Psi-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}.$$

(For now, keep e and A_μ untouched $\Rightarrow$ allowed in QED)

$$\mathcal{L}_{free} = \bar{\Psi}\left[i\gamma^\mu \partial_\mu - m\right]\Psi$$

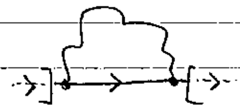
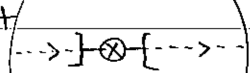
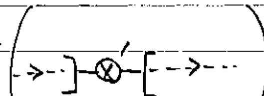
$$= Z_2 \bar{\Psi}_r \left[i\gamma^\mu\partial_\mu - m\right]\Psi_r$$

$$=: (1+\delta_{Z2})\,\bar{\Psi}_r\left[i\gamma^\mu\partial_\mu - (m_{pole}+\delta m)\right]\Psi_r$$

$$= \underbrace{\bar{\Psi}_r\left[i\gamma^\mu\partial_\mu - m_{pole}\right]\Psi_r}_{=\,\mathcal{L}_0} + \underbrace{\delta_{Z2}\,\bar{\Psi}_r i\gamma^\mu\partial_\mu\Psi_r - \left\{\delta_{Z2}\, m_{pole} + (1+\delta_{Z2})\delta m\right\}\bar{\Psi}_r\Psi_r}$$

treat as a part of $\mathcal{L}_{int}$

(along with $(1+\delta_{Z2})\,\bar{\Psi}_r\gamma^\mu\Psi_r A_\mu(-eQ)$)

Then $-i(A\not{p}+B)^{1\text{-loop}} =$  p^μ $+$ () $+$ ()

1-Loop $\Rightarrow \mathcal{O}(e^2)$ def. now

use $(-ieQ\gamma^\mu)$ rather than $(-i(1+\delta_{Z2})eQ\gamma^\mu)$

$i\delta_{Z2}\gamma^\mu p_\mu$

$-i\left\{\delta_{Z2}\, m_{pole} + \delta m\right\}$

We know already that $\delta_{Z2} \sim \mathcal{O}(e^2)$

To be more explicit,

$$\begin{cases} A^{(1)}(p^2, m^2) = \dfrac{\alpha Q^2}{4\pi}\displaystyle\int_0^1 dx \; -2(1-x)\ln\left(\dfrac{(1-x)\Lambda^2 + x m^2 - x(1-x)p^2}{+x m^2 - x(1-x)p^2}\right) \\ B^{(1)}(p^2, m^2) = \dfrac{\alpha Q^2}{4\pi}\displaystyle\int_0^1 dx \quad 4m \quad \ln\left(\dfrac{(1-x)\Lambda^2 + x m^2 - x(1-x)p^2}{+x m^2 - x(1-x)p^2}\right) \end{cases}$$

☆ $p^2 = m_{pole}^2$ must be a solution to

$$\boxed{\left(1 - A^{(1)}(p^2, m^2)\right)^2 p^2 = \left(m + B^{(1)}(p^2, m^2)\right)^2.}$$

Keeping terms of $\mathcal{O}(1)$ and $\mathcal{O}(\alpha)$ but not $\mathcal{O}(\alpha^2)$,

$$\boxed{m_{pole}^2 = m^2 + \left\{2m B^{(1)}(m^2, m^2) + 2m^2 A^{(1)}(m^2, m^2)\right\} + \mathcal{O}(\alpha^2)}$$

$$\boxed{m^2 = m_{pole}^2 - \left\{2m_{pole} B^{(1)}(m_{pole}^2, m_{pole}^2) + 2m_{pole}^2 A^{(1)}(m_{pole}^2, m_{pole}^2)\right\} + \mathcal{O}(\alpha^2)}$$

- When we keep track of the difference $(m^2 - m_{pole}^2)$ of order α, we can use m^2 and m_{pole}^2 interchangablly for the arguments of $A^{(1)}$ and $B^{(1)}$.
- $\begin{cases} m_{pole}^2 \text{ is determined by } m^2, \Lambda^2 \text{ and } e^2 \\ m^2 \text{ is determined by } m_{pole}^2, \Lambda^2 \text{ and } e^2 \end{cases}$

$\Rightarrow m = m_{pole} + (\delta m)$ expressed in terms of m_{pole}^2, Λ^2, e^2.

☆ The residue at the pole is determined by...

$$\frac{(1 - A(p^2, m^2))\not{p} + \cdots}{(1 - A(p^2, m^2))^2 p^2 - (m + B(p^2, m^2))^2} \xRightarrow{\text{residue}} \frac{(1 - A^{(1)})\big|}{(1 - A^{(1)})^2\big| - 2\dfrac{\partial A^{(1)}}{\partial p^2}\Big|(m_{pole}^2) - 2m\dfrac{\partial B^{(1)}}{\partial p^2}\Big|}$$

$$Z_2^{(1)} = (1 + \delta_{Z2}^{(1)}) = 1 + A^{(1)}(m_{pole}^2, m_{pole}^2) + 2m_{pole}^2\left(\frac{\partial A^{(1)}}{\partial p^2}\right)(m_{pole}^2, m_{pole}^2) + 2m_{pole}\frac{\partial B^{(1)}}{\partial p^2}(m_{pole}^2, m_{pole}^2)$$

We can use m^2 and m_{pole}^2 interchangablly in the arguments of $A^{(1)}$, $B^{(1)}$ etc., because we ignore $\mathcal{O}(\alpha^2)$ terms in δ_{Z2} in this calculation.

Then at $\mathcal{O}(\alpha)$, the 2-point function (correlator) becomes

$$\frac{i}{(\not{p} - m_{pole} + i\varepsilon) - \left[\not{p} A^{(1)}(p^2, m_{pole}) + B^{(1)}(p^2, m_{pole})\right] + \delta_{Z2}(m_{pole}, \Lambda, e)\not{p} - (\delta_{Z2} \cdot m_{pole} + \delta m)}$$

The coefficients of $\not{p}$ in the denominator : $\left[1 - A^{(1)}(p^2, m_{pole}^2) + \delta_{Z2}(m_{pole}, \Lambda, e)\right]$

$$-A^{(1)}(p^2, m_p^2) + A^{(1)}(m_p^2, m_p^2) = \frac{\alpha Q^2}{4\pi}\int_0^1 dx\ x(1-x)\left\{\ln\left(\frac{(1-x)\Lambda^2 + x m_p^2 - x(1-x)p^2}{+x m_p^2 - x(1-x)p^2}\right) - \ln\left(\frac{(1-x)\Lambda^2 + x m_p^2 - x(1-x)m_p^2}{+x m_p^2 - x(1-x)m_p^2}\right)\right\}$$

$\rightarrow$ $\boxed{\text{finite in the } \Lambda \to +\infty \text{ limit}}$

The quantum corrected correlator remains finite
when expressed in terms of observed (physical) parameter m_{pole}.

✓ The resulting expression does not depend on the choice of the regularization method ...

(strictly speaking, this is a desire/crietrion than a theorem.)

~~insensitive to the high-energy theory.~~

⟨ "divergence from infinitely many UV DOFs has been swept under the carpet (rug)" (in "m_{pole}") ⟩