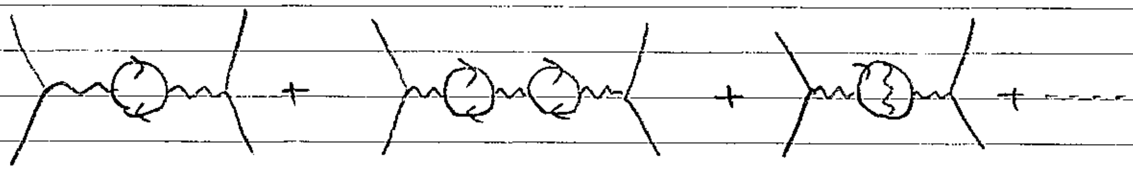
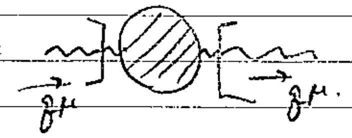
§5.2 QED in Renormalized Perturbation Theory

beyond electron self-energy



Σ(1 particle irreducible graphs) = 

$$=: i\left(q^2\eta_{\mu\nu} - q_\mu q_\nu\right)\Pi(q^2)$$

(non-trivial that the $q^2\eta_{\mu\nu}$ term and the $q_\mu q_\nu$ term are the same except sign. $\Leftarrow$ consequence of gauge sym)

Then (in the Feynman gauge)

$$\frac{-i\eta_{\mu\nu}}{(q^2+i\varepsilon)} + \frac{-i\delta_\mu{}^{\kappa_1}}{q^2}\, i\Pi(q^2)\left(q^2\eta_{\kappa_1\kappa_2} - q_{\kappa_1}q_{\kappa_2}\right)\frac{-i\delta^{\kappa_2}{}_\nu}{q^2}$$

$$+ \frac{-i\delta_\mu{}^{\kappa_1}}{q^2}\, i\Pi\left(q^2\eta_{\kappa_1\kappa_2} - q_{\kappa_1}q_{\kappa_2}\right)\frac{-i\eta^{\kappa_2\kappa_3}}{q^2}\, i\Pi\left(q^2\eta_{\kappa_3\kappa_4} - q_{\kappa_3}q_{\kappa_4}\right)\frac{-i\delta^{\kappa_4}{}_\nu}{q^2}$$

$$+ \cdots$$

$$= \frac{-i\eta_{\mu\nu}}{(q^2+i\varepsilon)(1-\Pi(q^2))}.$$ (the geometric series is summed up.)

At 1-loop level, just one graph contributes: .

Using the Pauli-Villars regularization (in unrenormalized fields & couplings)

$$\boxed{\Pi^{(1)}(q^2) = \frac{(eQ)^2}{2\pi^2}\int_0^1 dx\; x(1-x)\,\ln\left(\frac{M^2 - x(1-x)q^2}{\bar{M}_{reg}^2}\right)} \Leftarrow \text{homework VII}$$

- divergence under control (if $\bar{M}_{reg}^2 < \infty$)
- no mass shift; the pole remains @ $q^2=0$.
- only the normalization is affected.

- residue $\dfrac{-i\eta_{\mu\nu}Z_3}{q^2+i\varepsilon}$ @ $q^2=0$. $\Rightarrow$ $Z_3 = \dfrac{1}{1-\Pi(q^2)\big|_{q^2=0}} \simeq 1+\Pi^{(1)}(q^2)\big|_{q^2=0}$

$\Rightarrow$ The renormalized field $[A_\mu]_r := \dfrac{1}{\sqrt{Z_3}} A_\mu$

has the properly normalized two point correlation fcn.

$$\mathcal{L} \supset -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \Rightarrow -\frac{1}{4}Z_3[F_{\mu\nu}]_r[F^{\mu\nu}]_r \cong \underbrace{-\frac{1}{4}(F_{\mu\nu}F^{\mu\nu})_r}_{\mathcal{L}_0} \underbrace{-\frac{1}{4}\delta_{Z_3}^{(1)}(F_{\mu\nu}F^{\mu\nu})_r}_{\mathcal{L}_{int}}$$

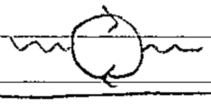
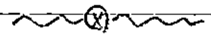
Feynman rule : $-i(q^2\eta_{\mu\nu}-q_\mu q_\nu)\delta_{Z_3}^{(1)}$

$= -i(q^2\eta_{\mu\nu}-q_\mu q_\nu)\Pi^{(1)}(q^2=0)$.

So, at the 1-loop level,

the sum of 1PI graphs for the renormalized photon field is

$$i(q^2\eta_{\mu\nu}-q_\mu q_\nu)\Pi^{(1)}(q^2) \quad + \quad -i(q^2\eta_{\mu\nu}-q_\mu q_\nu)\Pi^{(1)}(0)$$

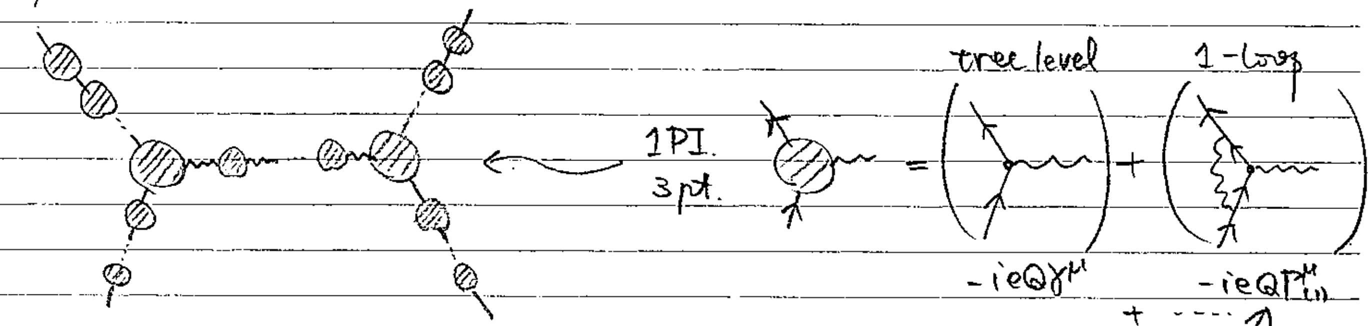
 

$$\Rightarrow \boxed{\underbrace{\Pi^{(1)}(q^2)-\Pi^{(1)}(0)}_{=\,\Pi^{(1)}_{ren}(q^2)} = \frac{(eQ)^2}{2\pi^2}\int_0^1 dx \; x(1-x)\ln\left(\frac{m_f^2-x(1-x)q^2}{m_f^2}\right)}$$

M_{reg}^2 : disappeared.

$\Pi(q^2)$ or $\Pi_{ren}(q^2)$ is called vacuum polarization.

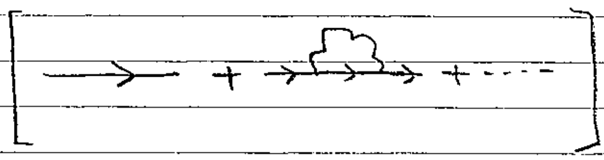
Finally, $\mathcal{L}_{QED} \supset -eQ(\bar{\Psi}\gamma^\mu A_\mu \Psi)$.



divergent.

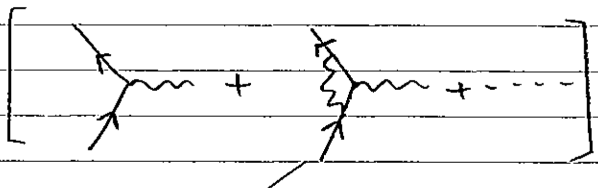
(finite log if Λ remains finite)

mass



the pole of the quantum corrected 2-point function $\Rightarrow$ observed mass $M \neq m_p$

coupling strength



the coupling strength of the cloud of $e^- + e^-(e^+e^-) + e^-(e^+e^-)^2 + \cdots$

$\Rightarrow$ measured strength $e_r \neq e$

$$-ieQ\Gamma^\mu_{(1)} =: -ieQ\left(\gamma^\mu V_1^{(1)}(q^2) + [\gamma^\mu, \gamma^\nu] q_\nu V_2^{(1)}(q^2) + \cdots\right)$$

Then the measured coupling strength should be ....

the three point function $\Psi_r - \bar{\Psi}_r - (A_\mu)_r$

$$= \prod_{\Psi_r, \bar{\Psi}_r, A_r} \left[\text{quantum corrected propagator}\right] \times \underbrace{Z_2\sqrt{Z_3} \times \left(-ieQ\gamma^\mu + (-ieQV_1^{(1)})\gamma^\mu + \cdots\right)}_{\parallel}$$

$$(-iQ\gamma^\mu)\left\{e + \left(eV_1^{(1)} + e\delta_{Z_2}^{(1)} + \frac{1}{2}e\delta_{Z_3}^{(1)}\right) + \cdots\right\}$$

So, $\boxed{e + \left(eV_1^{(1)}(q^2) + e\delta_{Z_2}^{(1)} + \frac{1}{2}e\delta_{Z_3}^{(1)}\right) + \mathcal{O}(e^5) = e_r(q^2)}$ $\leftarrow$ (eg. use the one for $q^2=0$.)

just like $\boxed{M^2 + \left(2M^2 A^{(1)}\big|_{p^2=M^2} + 2M B^{(1)}\big|_{p^2=M^2}\right) + \mathcal{O}(e^4) = m^2_{pole}}$

Lagrangian parameters $\longleftrightarrow$ observed parameters

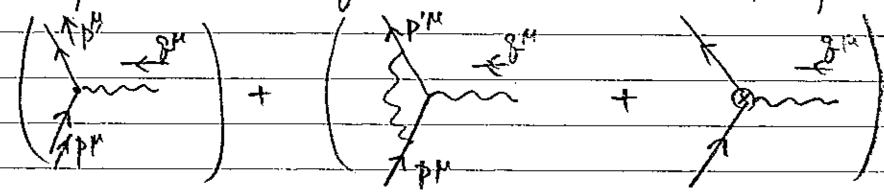
Renormalized perturbation theory for QED:

use $\{\Psi_r,\ A_\mu^{(r)},\ m_{pole},\ e_r\}$ for perturbation

$$\mathcal{L}_{QED} = \bar\Psi_r\left[i\gamma^\mu(\partial_\mu + i e_r Q A_{\mu,r}) - m_p\right]\Psi_r - \frac{1}{4}F^{(r)}_{\mu\nu}F^{(r)\mu\nu} \Rightarrow \mathcal{L}_0$$

$$\left.\begin{aligned} &+\delta_{Z2}\,\bar\Psi_r i\gamma^\mu\partial_\mu\Psi_r \qquad\qquad -\frac{1}{4}\delta_{Z3}F^{(r)}_{\mu\nu}F^{(r)\mu\nu} \\ &-(Z_2 M - m_p)\bar\Psi_r\Psi_r \\ &-Q(Z_2\sqrt{Z_3}\,e - e_r)(\bar\Psi_r\gamma^\mu A^{(r)}_\mu\Psi_r) \end{aligned}\right\}\Rightarrow \mathcal{L}_{int}$$

At 1-loop level, the quantum corrected 3pt fcn /(propagators)



$$= \left(\ \right) + \left(\ \ + \ \ \right)$$

$$-ie_rQ\gamma^\mu \qquad -ie_rQ\gamma^\mu V_1^{(1)}(q^2)+\cdots \qquad -iQ\left\{e-e_r+e_r\left(\delta^{(1)}_{Z2}+\frac{1}{2}\delta^{(1)}_{Z3}\right)\right\}$$

$$= -iQ\gamma^\mu\Big[e_r + \underbrace{e_r\left(V_1^{(1)}(q^2)-V_1^{(1)}(0)\right) + \left\{e-e_r+e_r\left(V_1^{(1)}(0)+\delta^{(1)}_{Z2}+\frac{1}{2}\delta^{(1)}_{Z3}\right)\right\}}_{\text{inserted}} + \mathcal{O}(e^5)\Big].$$

$\hookrightarrow \mathcal{O}(e^5)$

$\left(+[\gamma^\mu,\gamma^\nu]\text{ term}\right)$

~~Using the result~~

Peskin–Schroeder §6.3 (or QFT II week-15)

$$\Gamma^\mu_{(1)} = \frac{(eQ)^2}{16\pi^2}\int_\triangle dx\,dy\left[\gamma^\mu\cdot 2\left\{\ln\left(\frac{(1-w)\Lambda^2+w^2M^2-xyq^2}{w^2M^2-xyq^2}\right) + \frac{M^2\{1-4(1-w)+(1-w)^2\}+(1-x)(1-y)q^2}{w^2M^2-xyq^2}\right\} - \frac{[\gamma^\mu,\gamma^\nu]q_\nu}{4M}\,\frac{4M^2(1-w)w}{w^2M^2-xyq^2}\right],$$

$w := (x+y)$

we see that

$$V_1^{(1)}(q^2)-V_1^{(1)}(0) = \frac{\alpha Q^2}{2\pi}\int_\triangle dx\,dy\left[\ln\left(\frac{w^2m_p^2}{w^2m_p^2-xyq^2}\right)+\frac{m_p^2\{1-4(1-w)+(1-w)^2\}+(1-x)(1-y)q^2}{w^2m_p^2-xyq^2} - \frac{1-4(1-w)+(1-w)^2}{w^2}\right]$$

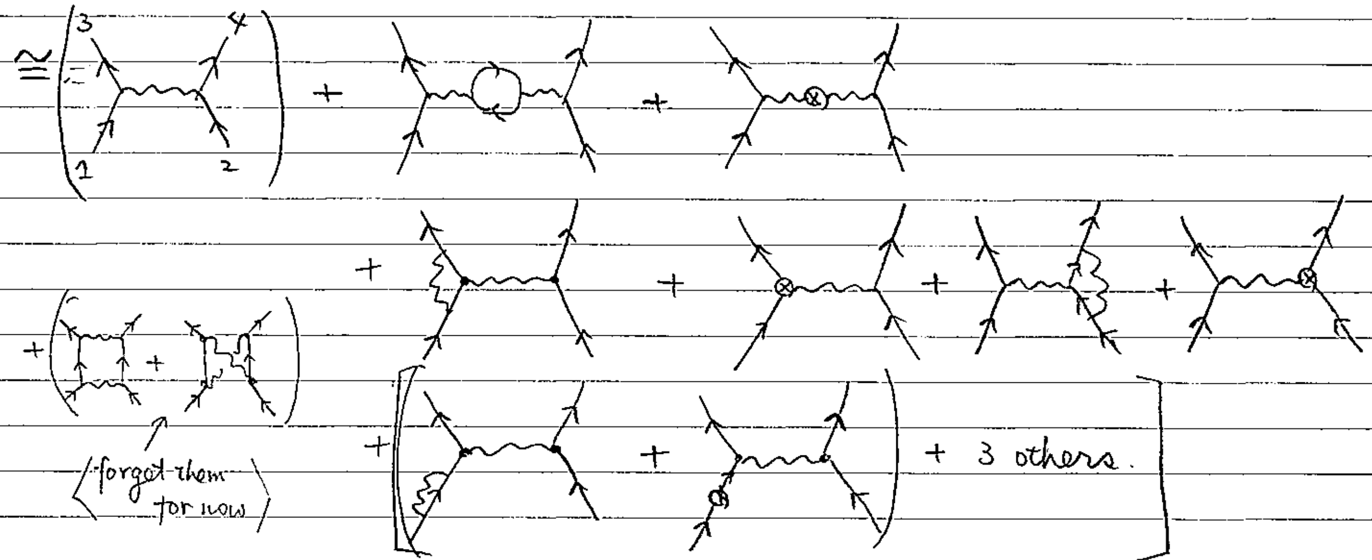
is free from the UV divergence (regulator scale Λ), but with quantum corrections. (not without IR divergence from $x+y\simeq 0$) PLUS

Recap At 1-loop level, the correlator of the renormalized fields

$$\int d^4x_3 \int d^4x_4\, e^{ip_3\cdot x_3} e^{ip_4\cdot x_4} \int d^4x_1 \int d^4x_2\, e^{-ip_1\cdot x_1} e^{-ip_2\cdot x_2}$$

$$\langle 0 | T\{ \Psi_{I,r}(x_4)\, \Psi_{I,r}(x_3)\, \bar\Psi_{I,r}(x_1)\, \bar\Psi_{I,r}(x_2) \exp\left[i\int d^4z\, \mathcal{L}_{\mathrm{int},I}(z)\right]\} | 0\rangle_{\text{connected}}.$$

$\cong$



$$= \Big[iQ^2 (e_r\gamma^\mu)\frac{\eta_{\mu\nu}}{q^2}(e_r\gamma^\nu) + iQ^2 (e_r\gamma^\mu)\frac{\eta_{\mu\nu}}{q^2}\left(\Pi^{(1)}(q^2)-\Pi^{(1)}(0)\right)(e_r\gamma^\nu)$$
$$+ iQ^2\, e_r\left(\Gamma^\mu_{(1)} - V_1^{(1)}(0)\gamma^\mu\right)\frac{\eta_{\mu\nu}}{q^2}(e_r\gamma^\nu)$$
$$+ iQ^2 (e_r\gamma^\mu)\frac{\eta_{\mu\nu}}{q^2}\, e_r\left(\Gamma^\nu_{(1)} - V_1^{(1)}(0)\gamma^\nu\right)\Big]$$

multiplied by $(2\pi)^4\delta^4(p_3+p_4-p_1-p_2)$ and quantum corrected propagators.

☆ Regularization parameters disappeared $(\Lambda, \bar M^2_{reg})$ when the correlator is expressed in terms of (e_r, m_{pole}) and we take the limit $\Lambda, \bar M^2_{reg} \to +\infty$.

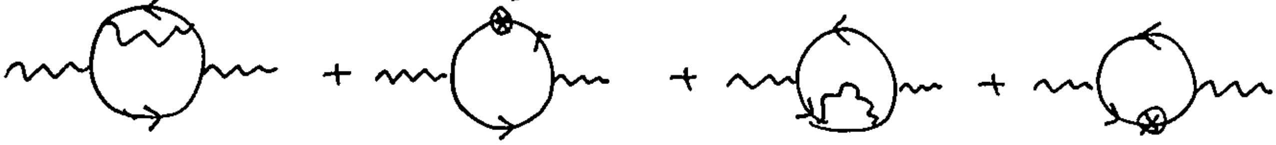
☆ Measurements look at "things" that include all sorts of quantum effects. Lagrangian parameters (e, M) are not the same as (e_r, m_{pole})

☆ The difference $(e - e_r)$ and $(M^2 - m^2_{pole})$ diverge if $\Lambda, \bar M^2_{reg}$ are literally ∞.

But the regulators are not more than an easy going version of some physical UV modification : the difference is finite. not scary.

$\boxed{\text{at 2-Loop}}$

$\delta_{22}^{(1)}, \delta_M^{(1)}$



$e-e_r+e_r\left(V_1^{(1)}(0)+\delta_{22}^{(1)}+\frac{1}{2}\delta_{23}^{(1)}\right)$

$\boxed{\delta_{23}^{(2)}}$

$\Uparrow$

$\boxed{-i(q^2\eta^{\mu\nu}-q^\mu q^\nu)\delta_{23}^{(2)}}$

$$= i\left(q^2\eta^{\mu\nu}-q^\mu q^\nu\right)\Pi_{ren}^{(2)}(q^2).$$

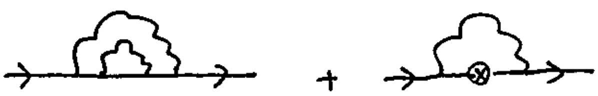
$\delta_{23}^{(2)}$: determined so that $\Pi_{ren}^{(2)}(q^2)=0$

The relative normalization of A_μ and $[A_\mu]_r$ should be such.

$\Updownarrow$

$$\frac{-i\eta^{\mu\nu}}{[q^2+i\varepsilon]\left(1-(\Pi_{ren}^{(1)}+\Pi_{ren}^{(2)})(q$$

has "Z_3" $=1$.



$\delta_{22}^{(1)}, \delta_M^{(1)}$

$(e-e_r+\cdots)$

$\boxed{\delta_{22}^{(2)}, \delta_M^{(2)}}$

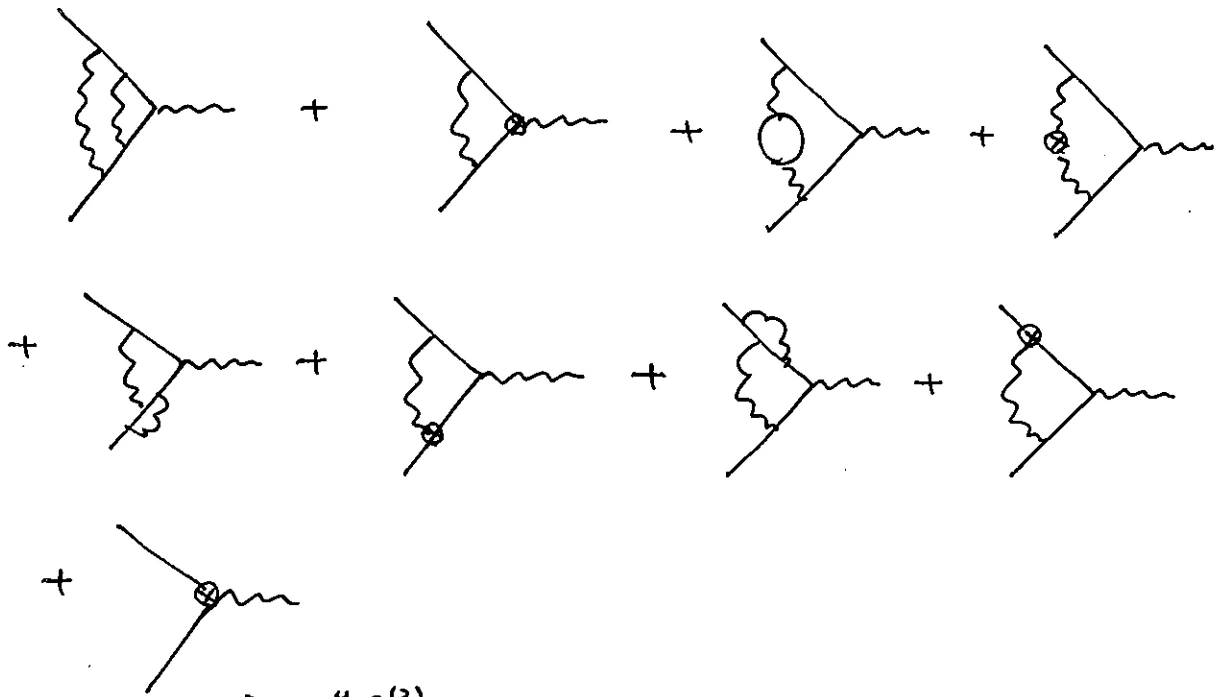
$\delta_{23}^{(1)}$

$$= -i\Sigma_{ren}^{(2)}(p,m)$$

$$= -i\left(A_{ren}^{(2)}\not p + B_{ren}^{(2)}\right)$$

$\delta_{22}^{(2)}$, $\delta_M^{(2)}$ determined so that

- $\boxed{\left(1-A_{ren}^{(1)}-A_{ren}^{(2)}\right)^2 m^2 = \left(m+B_{ren}^{(1)}+B_{ren}^{(2)}\right)^2\Big|_{p^2=m^2} \quad \mathrm{mod}\ \mathcal{O}(e^6)}$
- $\boxed{\left[\left(1-A_{ren}^{(1)}-A_{ren}^{(2)}\right)^2+2\left(A_{ren}^{(1)}+A_{ren}^{(2)}-1\right)\frac{\partial A_{ren}^{(1)+(2)}}{\partial p^2}-2\left(m+B_{ren}^{(1)+(2)}\right)\frac{\partial B^{(1)+(2)}}{\partial p^2}\right]\Bigg|_{p^2=m^2} = \left(1-A_{ren}^{(1)+(2)}\right)\Bigg|_{p^2=m^2}}$



$-Qie_r\gamma^\mu\,\underline{\underline{\delta_{Z1}^{(2)}}}$

determine so that $F_1(q^2)\big|_{q^2=0} = 1$

$$-Qie_r\,\Gamma^\mu_{ren} \equiv -Qie_r\left(F_1(q^2)\gamma^\mu - \frac{F_2(q^2)}{4m}[\gamma^\mu,\gamma^\nu]q_\nu\right)$$

$+ \text{mod}\,[\times(\not{p}-m)]$

$+ \text{mod}\,[(\not{p}'-m)\times]$

$\left\{\begin{array}{l} F_1 : \text{~~Pauli~~ Dirac} \\ F_2 : \text{Pauli} \end{array}\right.$

$\boxed{\text{Renormalization conditions. (on-shell)}}$

- ✓ Ψ_r canonically normalized. ("Z_2" = 1)
- ✓ fermion pole mass = m.
- ✓ $A_{r\mu}$ canonically normalized ("Z_3" = 1)
- ✓ $F_1(q^2=0) = 1$

$\Rightarrow$ determine. $\underbrace{\delta_{Z2}^{(n)},\ \delta_{Z3}^{(n)},\ (M-m),\ (e-e_r)\ \ (\text{or } \delta_{Z1}^{(n)})}_{\Uparrow}$

order by order.

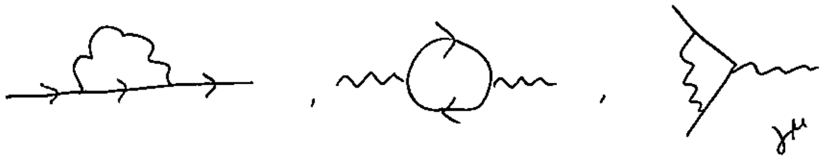
$\boxed{\text{counter terms.}}$

§5.3 Superficial Degree of Divergence

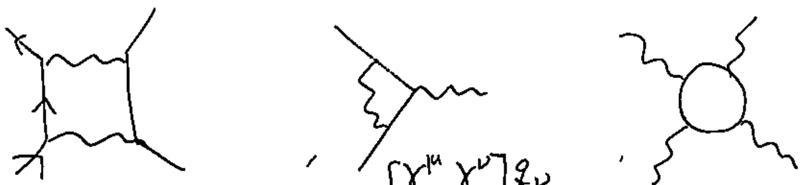
In QED (at 1-loop level)

4 renormalization conditions.

- e^- (and e^+) (pole) mass : m
- electric charge (at $q^\mu = 0$): e_r.
- Ψ_e field normalization : 1
- A_μ field : : 1. to rewrite $(M, e; \Lambda)$.



divergence appear. (taken care of).



+ ... any other amplitudes with divergence?

How do we know?

In QED, we've done all possible "renormalization".

fermion propagator

$$\frac{i[\not{p} + M]}{[p^2 - M^2 + i\varepsilon]} \Rightarrow -1$$

vector boson propagator (Feynman gauge)

$$\frac{-i\eta_{\mu\nu}}{(p^2 + i\varepsilon)} \Rightarrow -2.$$

scalar propagator

$$\frac{i}{[p^2 - M^2 + i\varepsilon]} \Rightarrow -2.$$

(scalar)^2 - vector vertex

$$\propto (p_1 - p_2)^\mu \Rightarrow +1.$$

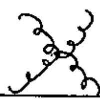
(fermion)^2 - vector vertex

$$\gamma^\mu \Rightarrow 0.$$

(vector)^3 vertex (non-Abelian)

 $\propto p^\mu \Rightarrow +1$

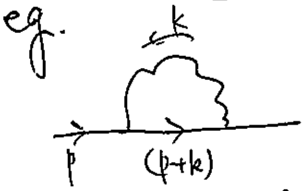
(vector)^4 vertex

 $\propto (\eta^{\mu\nu}\eta^{\rho\sigma} + \ldots) \Rightarrow 0.$

1-loop momentum $\dfrac{d^4 p}{(2\pi)^4} \Rightarrow +4$

$\boxed{\text{UV divergence: mass, external momenta don't matter.}}$

eg.

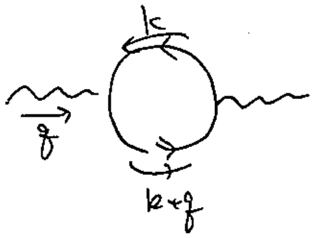


$\Rightarrow D = (-2) + (-1) + 4 = +1$ (γ, ψ)

truncate $\xrightarrow{\text{odd part.}}$ 0

$\int d^4k \; \gamma^\mu (\cancel{p}+\cancel{k}) \gamma_\mu \;\Rightarrow$ use $\cancel{p}$ not $\cancel{k}$

Logarithmic div.



$\Rightarrow D = (-1)^{\psi} \times 2 + 4 = \underline{\underline{+2}}$ for $(\;\;)_{\mu\nu}$.

but gauge symmetry ::

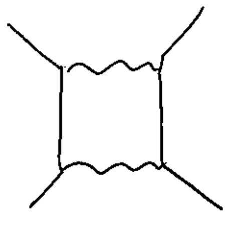
$$(\quad)_{\mu\nu} = i\left(\underline{q^2 \eta_{\mu\nu} - q_\mu q_\nu}\right)\Pi(q^2)$$

use q^2. or $q_\mu \cdot q_\nu$ not k^2 $\Rightarrow$ 0

Logarithmic. div

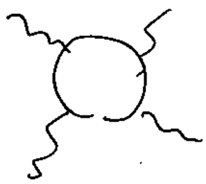
$\Rightarrow D = 0$. Logarithmic.

eg.



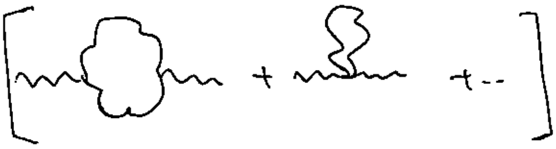
$\Rightarrow D = (-1)^{\psi} \times 2 + (-2)^{\gamma} \times 2 + 4 = -2$

$\Rightarrow D = 0$ .. but. $[\gamma^\mu, \gamma^\nu] \underline{q_\nu}$. $\Rightarrow -1$.



$D = 0$. but eventually $\Rightarrow -4$

no divergence



$\Rightarrow$ Logarithmic

$$D = 4\cdot(\#L) - I_\psi - 2I_\gamma$$

$$2\times(\#V) = 2I_\psi + E_\psi$$

$$(\#V) = 2I_\gamma + E_\gamma$$

$$D = 4\left[(\#C) + (\#I_{tot}) - (\#V)\right] - I_\psi - 2I_\gamma$$

$$= 4 + (3I_\psi + 2I_\gamma) - \left[\overset{4(\#V)}{\cancel{3(\#V) + (\#V)}}\right]$$

$$= \cancel{4 + (3I_\psi + 2I_\gamma) - \left[\left(3I_\psi + \tfrac{3}{2}E_\psi\right) + \left(2I_\gamma + E_\gamma\right)\right]} \quad \boxed{4 - \tfrac{3}{2}E_\psi - E_\gamma}$$

$$= 4 + \left(3(\#V) - \tfrac{3}{2}E_\psi\right) + \left((\#V) - E_\gamma\right) - 4(\#V)$$

for any higher loop amplitudes.

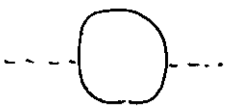
$D \le 4$, $E_\psi, E_\gamma > 0$: only finite amplitudes w/ $D > 0$.

All the $D \ge 0$ amplitudes exist in $\mathcal{L}_{QED}$. (w counterterms)

Yukawa theory

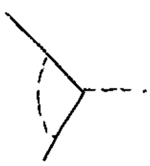
eg. $\mathcal{L} = \bar{\Psi}[i\gamma^\mu\partial_\mu - m]\Psi + \frac{1}{2}(\partial_\mu\phi)^2 - \frac{M^2}{2}\phi^2 - g\bar{\Psi}\Psi\phi$. ϕ: real scalar.

$$\boxed{D = 4 - \tfrac{3}{2}E_\psi - E_\phi}$$



$D=2$.

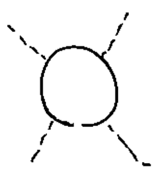
mass, wavefn ren. of ϕ.



$D=0$

$g + g_r$.

but



$D=0$.

need $\frac{\lambda}{4!}\phi^4$ term in $\mathcal{L}$.

otherwise, regulator scale remains.

(λ_r may just happen to be 0 though)

To recap

- particle species a : propagator p^{-k_a}
- vertex (interaction) type i : (field "a") $\times\, n_{ai}$ and $d_i \times (\partial_\mu)$

$$\boxed{D = 4L - \sum_a (k_a \cdot I_a) + \sum_i (d_i \cdot V_i)}$$

- for particle species "a" : $\boxed{(2I_a + E_a) = \sum_i n_{ai} V_i}$
- $\boxed{L - (C=1) = \left(\sum_a I_a\right) - \left(\sum_i V_i\right)}$

$$\Rightarrow D = 4 + \sum_a (4-k_a) I_a + \sum_i (d_i - 4) V_i$$

$$= 4 + \sum_i \Big(\underbrace{\sum_{a} \left(\frac{4-k_a}{2}\right) n_{ai} + d_i}_{\Delta_i} - 4 \Big) V_i - \sum_a \left(\frac{4-k_a}{2}\right) E_a .$$

- $\Delta_i \equiv \sum_a \left(\frac{4-k_a}{2}\right) n_{ai} + d_i$ (naive) operator dim. (eg)

$\boxed{(4-\Delta_i) : \text{mass-dimension of the coefficient of the vertex "}i\text{"}}$

~~scalar~~ $\left(\frac{4-k_a}{2}\right)$: (naive) mass-dim. of field "a"

- scalar : $\frac{i}{p^2-m^2+i\varepsilon} \Rightarrow 1$
- vector : $\frac{-i\eta_{\mu\nu}}{p^2+i\varepsilon} \Rightarrow 1$. Ψ : $\frac{i(\not{p}+m)}{p^2-m^2} \Rightarrow 3/2$.

$\boxed{\text{QED, (Yukawa} + \phi^4\text{) theory}}$. Both have only interactions where $\Delta_i = 4$. $(4-\Delta_i = 0)$

$\Rightarrow$ limited variety of amplitudes ($\{E_a\}$) where $D \geq 0$ (divergent).

$\boxed{\text{If all the interactions of a theory satisfy } \Delta_i \leq 4. \ (4-\Delta_i \geq 0).}$

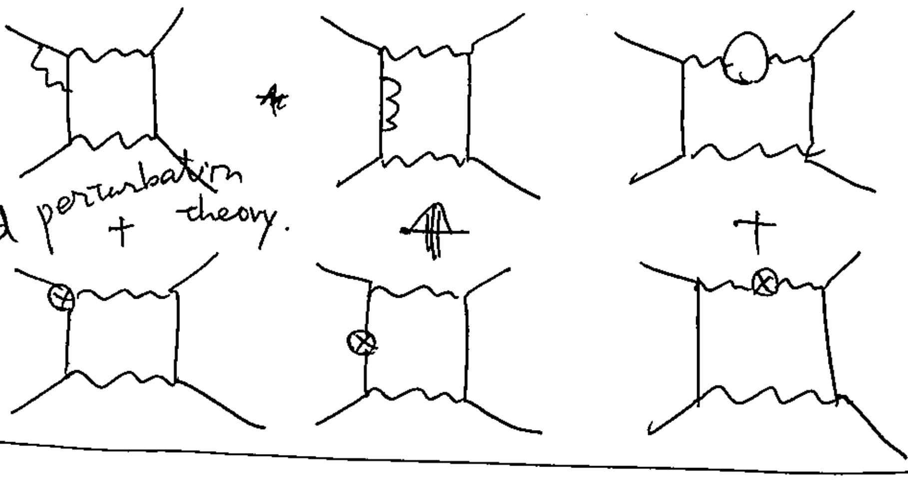
Renormalizable QFT

If all possibly divergent amplitudes have corresponding terms in $\mathcal{L}$ ~~so that~~ ($\Leftrightarrow$ $\exists$ counter terms),
such amplitudes can be written in terms of observed (renormalized) coupling coefficients (and kinematical variables)
w/o referring to the regulator scale.

subtlety 1

subdiagram may diverge even when $D<0$.

eg.



but in renormalized perturbation theory.

subtlety 2

$\exists$ counter term alone is not enough.

$$\text{eg. } \Lambda^2 \ln\left(\frac{\Lambda^2}{p^2}\right) - \underset{\text{c.t.}}{\Lambda^2 \ln\left(\frac{\Lambda^2}{m^2}\right)} \Rightarrow \Lambda^2 \ln\left(\frac{m^2}{p^2}\right)$$

finite at the kinematics for the renormalization condition.

Bogoliubov – Parasiuk, Hepp, Zimmermann.

renormalizable