§5.3 Superficial Degree of Divergence

In QED (at 1-loop level)

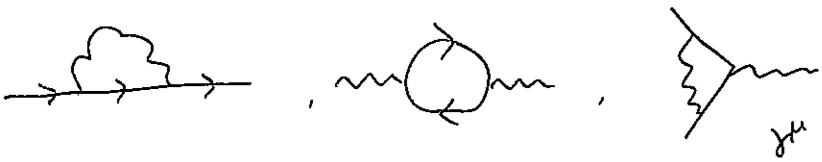
4 renormalization conditions.

e^- (and e^+) (pole) mass : m

electric charge (at $q^\mu = 0$): e_r

Ψ_e field normalization : 1

A_μ field : : 1. to rewrite $(M, e; \Lambda)$.



γ^μ

divergence appear. (taken care of).

$[\gamma^\mu, \gamma^\nu] q_\nu$

+ ... any other amplitudes with divergence?

How do we know?

In QED, we've done all possible "renormalization".

fermion propagator

$$\frac{i(\not{p}+M)}{[p^2 - M^2 + i\varepsilon]} \Rightarrow -1$$

vector boson propagator (Feynman gauge)

$$\frac{-i\eta_{\mu\nu}}{(p^2+i\varepsilon)} \Rightarrow -2.$$

scalar propagator

$$\frac{i}{[p^2 - M^2 + i\varepsilon]} \Rightarrow -2.$$

(scalar)^2 - vector vertex

$\propto (p_1 - p_2)^\mu \Rightarrow +1.$

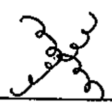
(fermion)^2 - vector vertex

$\gamma^\mu \Rightarrow 0.$

(vector)^3 vertex (non-Abelian)

$\propto p^\mu \Rightarrow +1$

(vector)^4 vertex

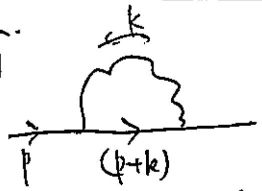


$\propto (\eta^{\mu\nu}\eta^{\rho\sigma} + \dots) \Rightarrow 0.$

1-loop momentum $\frac{d^4 p}{(2\pi)^4} \Rightarrow +4$

$\boxed{\text{UV divergence: mass, external momenta don't matter.}}$

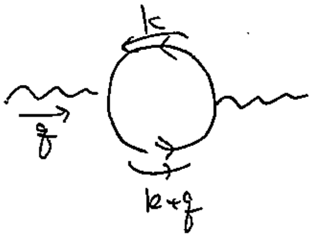
eg.



$\Rightarrow D = \overset{\gamma}{(-2)} + \overset{\psi}{(-1)} + 4 = +1$ truncate $\xrightarrow{\text{odd part.}}$ 0

$\int d^4k\ \gamma^\mu (\not p + \not k) \gamma_\mu \Rightarrow$ use $\not p$ not $\not k$

logarithmic div.



$\Rightarrow D = \overset{\psi}{(-1)} \times 2 + 4 = +2$ for $(\ \ )_{\mu\nu}$.

but gauge symmetry:

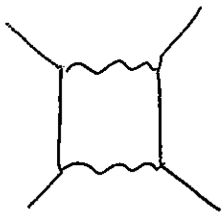
$(\ \ \ )_{\mu\nu} = i(q^2 \eta_{\mu\nu} - q_\mu q_\nu)\,\Pi(q^2)$

use q^2. or $q_\mu \cdot q_\nu$ not k^2 $\Rightarrow 0$

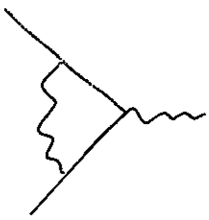
logarithmic. div

$\Rightarrow D = 0$. logarithmic.

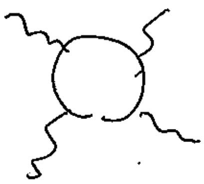
eg.



$\Rightarrow D = \overset{\psi}{(-1)} \times 2 + \overset{\gamma}{(-2)} \times 2 + 4 = -2$



$\Rightarrow D = 0$ .. but. $[\gamma^\mu, \gamma^\nu]\, q_\nu$. $\Rightarrow -1$..



$D = 0$. but eventually $\Rightarrow -4$

no divergence

[ +  + ..] $\Rightarrow$ logarithmic

$$D = 4\cdot(\#L) - I_\psi - 2I_\gamma$$

$$2\times(\#V) = 2I_\psi + E_\psi$$

$$(\#V) = 2I_\gamma + E_\gamma$$

$$D = 4\left[(\#C) + (\#I_{tot}) - (\#V)\right] - I_\psi - 2I_\gamma$$

$$= 4 + (3I_\psi + 2I_\gamma) - \left[\cancel{3(\#V) + (\#V)}\right] \quad 4(\#V)$$

$$= \cancel{4 + (3I_\psi + 2I_\gamma) - \left[(3I_\psi + \tfrac{3}{2}E_\psi) + (2I_\gamma + E_\gamma)\right]} \quad \boxed{4 - \tfrac{3}{2}E_\psi - E_\gamma}$$

$$= 4 + \left(3(\#V) - \tfrac{3}{2}E_\psi\right) + \left((\#V) - E_\gamma\right) - 4(\#V)$$

for any higher loop amplitudes.

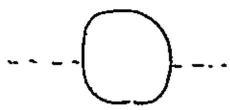
$D \le 4$, $E_\psi, E_\gamma > 0$: only finite amplitudes w/ $D>0$

All the $D \ge 0$ amplitudes exist in $\mathcal{L}_{QED}$ (w counter terms)

Yukawa theory

eg. $\mathcal{L} = \bar{\Psi}[i\gamma^\mu \partial_\mu - m]\Psi + \frac{1}{2}(\partial_\mu\phi)^2 - \frac{M^2}{2}\phi^2 - g\bar{\Psi}\Psi\phi$ ϕ: real scalar.

$$\boxed{D = 4 - \tfrac{3}{2}E_\psi - E_\phi}$$



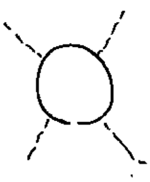
$D=2$.

mass, wavefn ren. of ϕ.

$D=0$

$g + g_r$.

but



$D=0$.

need $\frac{\lambda}{4!}\phi^4$ term in $\mathcal{L}$.

otherwise, regulator scale remains.

(λ_r may just happen to be 0 though)

To recap

- particle species a : propagator p^{-k_a}
- vertex (interaction) type i : (field "a") $\times n_{ai}$ and $d_i \times (\partial_\mu)$

$$\boxed{D = 4L - \sum_a (k_a \cdot I_a) + \sum_i (d_i \cdot V_i)}$$

- for particle species "a" : $\boxed{(2I_a + E_a) = \sum_i n_{ai} V_i}$
- $\boxed{L - (C=1) = (\sum_a I_a) - (\sum_i V_i)}$

$$\Rightarrow D = 4 + \sum_a (4-k_a) I_a + \sum_i (d_i - 4) V_i$$

$$= 4 + \sum_i \Big(\underbrace{\sum_a \left(\tfrac{4-k_a}{2}\right) n_{ai} + d_i}_{\Delta_i} - 4 \Big) V_i - \sum_a \left(\tfrac{4-k_a}{2}\right) E_a .$$

- $\Delta_i \equiv \sum_a \left(\frac{4-k_a}{2}\right) n_{ai} + d_i$ (naive) operator dim. (eg)

$\boxed{(4-\Delta_i) : \text{ mass-dimension of the coefficient.}}$ of the vertex "i"

~~scalar~~ $\left(\frac{4-k_a}{2}\right)$: (naive) mass-dim. of field "a"

scalar : $\dfrac{i}{p^2 - m^2 + i\varepsilon}$ $\Rightarrow 1$

vector : $\dfrac{-i\eta_{\mu\nu}}{p^2 + i\varepsilon}$ $\Rightarrow 1$. Ψ : $\dfrac{i(\not{p}+m)}{p^2-m^2} \Rightarrow 3/2$.

$\boxed{\text{QED, (Yukawa} + \phi^4\text{) theory}}$.

Both have only interactions where $\Delta_i = 4$ $(4-\Delta_i = 0)$

$\Rightarrow$ limited variety of amplitudes ($\{E_a\}$) where $D \geq 0$. (divergent).

$$\boxed{\begin{array}{l}\text{If all the interactions of a theory} \\ \quad \text{satisfy } \Delta_i \leq 4. \ (4-\Delta_i \geq 0).\end{array}}$$

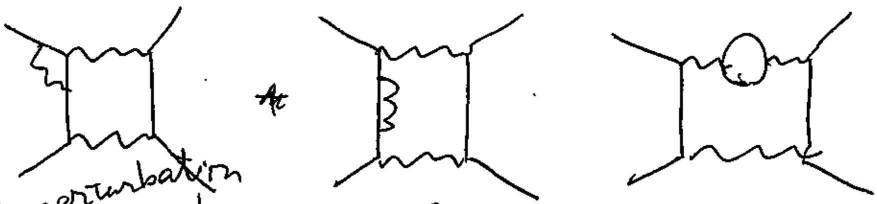
Renormalizable QFT

If all possibly divergent amplitudes have corresponding terms in $\mathcal{L}$ ~~so that~~ ($\Leftrightarrow$ $\exists$ counter terms), such amplitudes can be written in terms of observed (renormalized) coupling coefficients (and kinematical variables) w/o referring to the regulator scale.

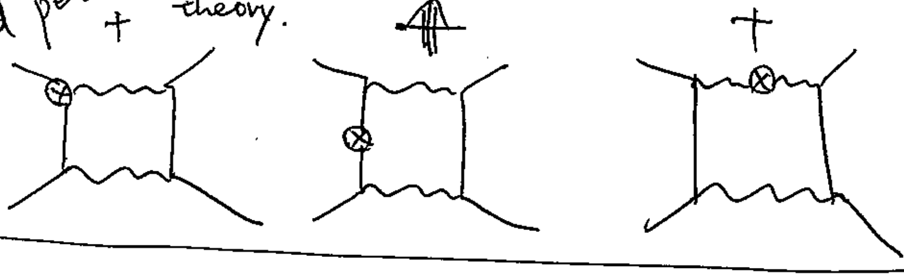
subtlety 1

subdiagram may diverge even when $D<0$.

eg.



but in renormalized perturbation theory



subtlety 2

$\exists$ counter term alone is not enough.

$$\text{eg.}\quad \Lambda^2 \ln\left(\frac{\Lambda^2}{p^2}\right) - \underbrace{\Lambda^2 \ln\left(\frac{\Lambda^2}{M^2}\right)}_{\text{c.t.}} \Rightarrow \Lambda^2 \ln\left(\frac{M^2}{p^2}\right)$$

finite at the kinematics for the renormalization condition.

Bogoliubov – Parasiuk, Hepp, Zimmermann.

renormalizable

§ 5.4 Renormalized Perturbation Theory of "Non-Renormalizable" Theories.

Historically

Renormalizable theories : Renormalizable QFT's. which need only a finite # of renormalized coefficients.

$$\boxed{D = 4 - \left(\frac{3}{2}E_\psi + E_\phi + E_A\right) + \sum_i (\Delta_i - 4) V_i}$$

Δ_i : (naive) operator dim.

✓ If $(\Delta_i - 4) > 0$ in one of interaction terms..

$D > 0$ unlimitedly.

✓ ~~set~~ infinite # of renormalization conditions.

what's wrong? divergent ⇒ subtract by counter terms.

$$\underset{\substack{\uparrow \\ \infty \text{ many}}}{(g_i)_r} = \text{fn}_i\big(\underset{\substack{\uparrow \\ \text{many}}}{\{g_k\text{'s}\}}\, ; \Lambda\big).$$

doable? well-defined?

✓ In the real world..

ν mass may be due to $\mathcal{L}_{int} = \frac{1}{M}\, \text{"}\overbrace{\Psi\Psi}^{\text{lepton}}\overbrace{\phi\phi}^{\text{Higgs field}}\text{"}$

$\Delta = 5$ dimension-5 operator.

☆ matrix elements $\mathcal{M}$

$\mathcal{S}$-matrix : mass-dim $= -(\#\mathrm{Ext}) \Leftarrow \langle \vec{p} | \vec{q} \rangle = \underbrace{(2\pi)^3 \delta^3(\vec{p}-\vec{q})(2E_{\vec{p}})}_{\text{mass-dim} = -2}$

$|\vec{q}\rangle^{in}$: mass-dim $= -1$.

$$\mathcal{S} = (2\pi)^4 \delta^4(p_{in} - p_{out})\, i\mathcal{M} + \mathbb{1}.$$

$$\Rightarrow \mathcal{M} : \text{mass-dim} = 4 - (\#\mathrm{Ext})$$

☆ Coefficients w/ mass dimensions

$$\begin{cases} \mathcal{L}_{int} \supset m^{(4-\Delta_j)}\, \mathcal{O}_j & \text{renormalizable operators } (4-\Delta_j \geq 0) \\ \mathcal{L}_{int} \supset \frac{1}{M^{\Delta_i - 4}}\, \mathcal{O}_i & \text{non-renormalizable operators } (\Delta_i - 4 > 0) \end{cases}$$

Think of a theory whose non-ren. operators come with coefficients scaled by a common energy scale M.

Require a precision (for a fixed δ)

$$\mathcal{M} \sim (\text{Energy})^{4-(\#\mathrm{Ext})} \left[1 + \ldots \left(\frac{\text{Energy}}{M}\right) + \cdots \underbrace{+ \ldots \left(\frac{\text{Energy}}{M}\right)^{\delta}}_{(*)} + \mathcal{O}\left(\left(\frac{\text{Energy}}{M}\right)^{\delta+1}\right) \right]$$

for various processes.

Then only finitely many operators contribute. (for a fixed δ)

The renormalized coupling constants of those operators are set so that the renormalization conditions at some energy scale are statisfied.

$\boxed{\text{Log corrections appearing in } (*) \text{ provide non-trivial predictions.}}$

example : Lagrangian of pions.