☆ matrix elements $\mathcal{M}$

S-matrix : mass-dim $= -(\text{Ext})$ $\Leftarrow$ $\underbrace{\langle \vec{p} | \vec{q} \rangle = (2\pi)^3 \delta^3(\vec{p}-\vec{q})(2E_{\vec{p}})}_{\text{mass-dim}=-2}$

$|\vec{q}\rangle^{in}$: mass-dim $= -1$.

$S = (2\pi)^4 \delta^4(p_{in} - p_{out})\, i\mathcal{M} + \mathbb{1}$.

$\Rightarrow$ $\mathcal{M}$: mass-dim $= 4 - (\text{Ext})$

☆ Coefficients w/ mass dimensions

$$\begin{cases} \mathcal{L}_{int} \supset m^{(4-\Delta_j)} \mathcal{O}_j & \text{renormalizable operators} \quad (4-\Delta_j \geq 0) \\ \mathcal{L}_{int} \supset \frac{1}{M^{\Delta_i - 4}} \mathcal{O}_i & \text{non-renormalizable operators} \quad (\Delta_i - 4 > 0) \end{cases}$$

Think of a theory whose non-ren. operators come with coefficients scaled by a common energy scale M.

Require a precision (for a fixed δ)

$$\mathcal{M} \sim (\text{Energy})^{4-(\text{Ext})} \left[1 + \cdots \left(\frac{\text{Energy}}{M}\right) + \cdots + \underbrace{\cdots \left(\frac{\text{Energy}}{M}\right)^{\delta}}_{(*)} + \mathcal{O}\left(\left(\frac{\text{Energy}}{M}\right)^{\delta+1}\right) \right]$$

for various processes.

Then only finitely many operators contribute. (for a fixed δ)

The renormalized coupling constants of those operators are set so that the renormalization conditions at some energy scale are statisfied.

$\boxed{\text{Log corrections appearing in } (*) \text{ provide non-trivial predictions.}}$

example : Lagrangian of pions.

§6. Renormalization Group.

§6.1. Variations of Renormalization Conditions.

For QED, we chose (in §5)

- ✓ $\langle \psi\, \bar\psi \rangle$ pole mass : $\boxed{m}$
- ✓ $\psi - \bar\psi - A_\mu$ coupling (electric charge) at $q_\mu = 0$: $\boxed{e_r}$
- ✓ $\langle \psi\, \bar\psi \rangle$ canonical normalization at $p^2 = m^2$.
- ✓ $\langle A_\mu A_\nu \rangle$ 〃 〃 at $p^2 = 0$ (pole, on-shell)

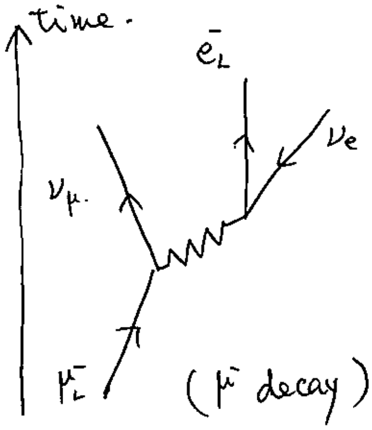
but not unique ....

eg. 1 Electroweak theory. $SU_L(2) \times U(1)_Y \to U(1)_{QED}$.
symmetry breaking by a scalar field condensation.

parameters

$$\left[\begin{array}{l} g,\ g' \ (\text{coupling constants of } SU(2)_L \times U(1)_Y) \\ \langle \phi \rangle = v/\sqrt{2}. \end{array}\right.$$

observables $\quad \alpha_{QED} = (e^2/4\pi)$, m_Z, m_W, $G_\mu(\mu \to \nu_\mu + e + \bar\nu_e)$



(μ^- decay)

at tree level.

$$m_W = g\frac{v}{2}, \quad m_Z = \sqrt{g^2+g'^2}\,\frac{v}{2}, \quad e = \frac{g g'}{\sqrt{g^2+g'^2}}, \quad \frac{G_\mu}{\sqrt{2}} = \frac{g^2}{8 m_W^2}$$

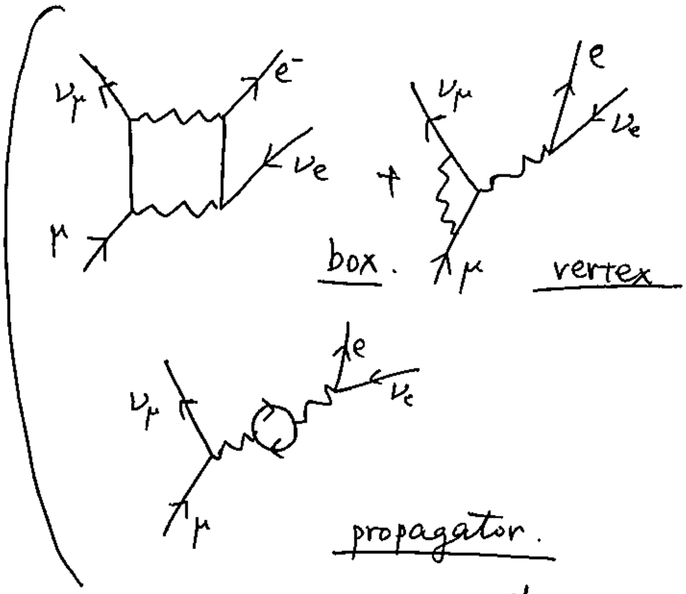
$$\Rightarrow \boxed{\frac{G_\mu}{\sqrt{2}} = \frac{\pi\,\alpha_{QED}}{2 m_W^2\left(1 - \frac{m_W^2}{m_Z^2}\right)}}$$

(homework)

At 1-Loop level.

$$\frac{\pi\alpha_{QED}}{2m_W^2\left(1-\frac{m_Z^2}{m_W^2}\right)} \neq \frac{G_\mu}{\sqrt{2}}$$

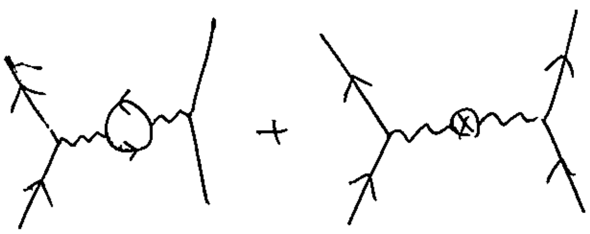
$$= \frac{\pi\alpha_{QED}}{2m_W^2\left(1-\frac{m_Z^2}{m_W^2}\right)(1-\Delta r)}$$



quantum correction to $(\mu^- \to \nu_\mu e^- \bar{\nu}_e)$

use 3 observables to fix the renormalized values of the theory parameters. g, g', v.

eg. 2 scattering at large momentum transfer.



$$\frac{-i\eta^{\mu\nu}}{q^2+i\varepsilon}\Big[\underset{\text{tree}}{1} + \underset{\text{1-Loop}}{\Pi^{(1)}_{ren}(q^2)} + \underset{\text{2-Loop.}}{\cdots}\Big]$$

$$\Pi^{(1)}_{ren}(q^2) = \Pi^{(1)}(q^2) - \Pi^{(1)}(q^2=0) = \frac{e^2}{2\pi^2}\int_0^1 dx\, x(1-x)\left\{\ln\left(\frac{m_e^2-x(1-x)q^2}{M_{reg}^2}\right) - \ln\left(\frac{m_e^2}{M_{reg}^2}\right)\right\}$$

fixed order

$$\left[1+\Pi^{(1)}_{ren}(q_0^2)\right] \neq \frac{1}{\left[1-\Pi^{(1)}_{ren}(q_0^2)\right]} \quad \text{if} \quad \frac{\alpha_{QED}}{\pi}\ln\left(\frac{(-q)^2}{m_e^2}\right) \not\ll 1$$

$\Rightarrow$ ~~why not using~~

$$\cancel{\langle A_r^\mu(q)\, A_r^\nu(-q)\rangle = \frac{-i\eta^{\mu\nu}}{[q^2+i\varepsilon][1-\Pi^{(1)}_{ren}(q_0^2)]} \sim \frac{-i\eta^{\mu\nu}}{[q^2+i\varepsilon][1-\Pi^{(1)}_{ren}(q^2)]}}$$

when $q^2 \sim q_0^2$. ?

remember

$$\langle \{A_*^\mu(q)\ A_*^\nu(-q)\}\rangle \sim \frac{-i\eta^{\mu\nu}}{[q^2+i\varepsilon]\,[1-\Pi^{(1)}(q^2)]}$$

↳ large log quant. corr.

$[1-\Pi^{(1)}(0)]\times$ ⇓

$$\langle \{A_r^\mu(q)\ A_r^\nu(-q)\}\rangle \sim \frac{-i\eta^{\mu\nu}[1-\Pi^{(1)}(q^2=0)]}{[q^2+i\varepsilon]\,[1-\Pi^{(1)}(q^2)]} \sim \frac{-i\eta^{\mu\nu}}{[q^2+i\varepsilon]\,[1-\underbrace{[\Pi^{(1)}(q^2)-\Pi^{(1)}(0)]}_{\equiv\ \Pi^{(1)}_{ren}(q^2)}]}$$

why not $\times\ [1-\Pi^{(1)}(q^2=-\vec{q}_0^{\,2})]$ instead?

$$\begin{cases} A_*^\mu = A_r^\nu\sqrt{Z_3} = A_r^\nu \dfrac{1}{\sqrt{1-\Pi(0)}} \\ A^\nu = A^\nu_{r(q_0)}\sqrt{Z_3^{(q_0)}} = A^\nu_{r(q_0)}\dfrac{1}{\sqrt{1-\Pi(-\vec{q}_0^{\,2})}} \end{cases} \qquad \left(A^\nu_{r(q_0)} = \sqrt{\frac{Z_3}{Z_3^{(q_0)}}}\,A_r^\nu\right)$$

$$\langle \{A^\mu_{r(q_0)}(q)\ A^\nu_{r(q_0)}(-q)\}\rangle \sim \frac{-i\eta^{\mu\nu}[1-\Pi^{(1)}(-|\vec{q}_0|^2)]}{[q^2+i\varepsilon]\,[1-\Pi^{(1)}(q^2)]} \sim \frac{-i\eta_{\mu\nu}}{[q^2]\left[1-\Pi^{(1)}_{ren}(q^2)-\Pi^{(1)}_{ren}(-|\vec{q}_0|^2)\right]}$$

- loop expansion in perturbation with $A^\mu_{r(q_0)}$

  $: 1+[\Pi^{(1)}_{ren}(q^2)-\Pi^{(1)}_{ren}(\vec{q}_0^{\,2})]+[\quad]^2+\cdots$

  $\Rightarrow$ good efficient expansion if $|q^2|\sim|-|\vec{q}_0|^2|$

- finitely different $\begin{bmatrix}\text{counterterm}\\ \text{field normalization}\end{bmatrix}$

only cost to pay — good enough renormalization condition.

$$\langle\Omega|T\{\phi_{r(q_0)}(p)\ \phi_{r(q_0)}(-p)\}|\Omega\rangle \sim \frac{i\,(Z/Z^{(q_0)})}{p^2-m^2+i\varepsilon}$$

$Z/Z^{(q_0)}$ not necessarily 1 $\Rightarrow$ $\underbrace{\times\prod_i\left(Z_i/Z_i^{(q_0)}\right)^{1/2}}_{\text{do not forget to}}$ for S-matrix. to amputated amplitudes

§ 6.2 Renormalization at Energy Scale "E".

$(\gamma\gamma)$

$$\boxed{\Sigma^{(1)}(q^2; M_{reg}^2) = \frac{e^2}{2\pi^2}\int_0^1 dx\; x(1-x)\; \ln\left(\frac{m_e^2 - x(1-x)q^2}{M_{reg}^2}\right).}$$

$(\psi\bar\psi)$

$$\boxed{\Sigma^{(1)}(p; \Lambda^2) = \frac{e^2}{16\pi^2}\int_0^1 dx\left[-2\not p(1-x) + 4m_e\right]\ln\left(\frac{(1-x)\Lambda^2 + x m_e^2 - x(1-x)p^2}{x m_e^2 - x(1-x)p^2}\right)}$$

$(\psi\bar\psi A)$

$$ie\Gamma^\mu_{(1)} \cong (ie)\frac{e^2}{16\pi^2}\int dx\,dy \left(2\gamma^\mu\left[\ln\left(\frac{(1-x-y)\Lambda^2 + (x+y)^2 m_e^2 - xyq^2}{(x+y)^2 m_e^2 - xyq^2}\right) + \frac{\left[m^2\{1-4(1-x-y)+(1-x-y)^2\} + (1-x)(1-y)q^2\right]}{\{(x+y)^2 m_e^2 - xyq^2\}}\right] - [\gamma^\mu, \gamma^\nu]\,q_\nu\,\frac{(1-x-y)(x+y)m_e}{\{(x+y)^2 m_e^2 - xyq^2\}}\right)$$

$$\Sigma^{(1)}_{ren}(q^2) \equiv \Sigma^{(1)}(q^2; M_{reg}^2) - \Sigma^{(1)}(\underline{q^2=0}; M_{reg}^2) \sim \ln\left(\frac{m^2}{m^2 - x(1-x)q^2}\right) \quad [q^2=0]$$

$$\Sigma^{(1)}(p;\Lambda) - \Sigma^{(1)}(\underline{p^2=m_e^2};\Lambda) \sim \ln\left(\frac{x m_e^2 - x(1-x)m_e^2}{x m_e^2 - x(1-x)p^2}\right) \quad [p^2\to m_e^2]$$

$$\Gamma^\mu_{(1)}(q;\Lambda) - \Gamma^\mu_{(1)}(\underline{q^2=0};\Lambda) \sim 2\gamma^\mu \ln\left(\frac{(x+y)^2 m_e^2 - xy\times 0}{(x+y)^2 m_e^2 - xyq^2}\right). \quad q^2=0$$

$\Rightarrow$ $\boxed{\text{always leave } \frac{\alpha_e}{\pi}\ln\left(\frac{E^2}{q^2}\right) \text{ quantum corrections.}}$

(E: energy scale of renormalization conditions.)

if we choose $E^2 \sim 0$ or m_e^2 ...
perturbative expansion is not efficient when $m_e^2 \ll |q^2|$.

$\Rightarrow$ $\boxed{\text{renomalize at energy scale } \underline{E}}$ (often use μ instead.)

different renormalization condition

$\Rightarrow$ different values for the renormalized coupling constants.

eg. QED $(\psi\bar{\psi}A)$ coupling.

$$i e_r \gamma^\mu \ \left(+\ i e_r \Gamma^\mu_{(1).\mathrm{ren}}\right) \Longrightarrow \left(i e_r \gamma^\mu + i e_r \Gamma^\mu_{(1).\mathrm{ren}}\right) \times \frac{1}{\sqrt{1-A_{\mathrm{ren}}(p^2=-\mu^2)}^{\,2}\sqrt{1-\Pi_{\mathrm{ren}}}}$$

coupling at $q^\mu = 0$

vanish at $q^\mu = 0$

field : residue = 1 on-shell

coupling at $[q^2 = -\mu^2]$

$(q^2 = -\mu^2)$

field residue = 1 at $(-p^2 = \mu^2 \ / \ -q^2 = \mu^2.)$

(*) $\Rightarrow$

$$\boxed{\begin{aligned} i e_r(\mu) \simeq i e_r & \left\{ 1 + \frac{e_r^2}{16\pi^2}\int dx\,dy \ \ 2\ln\left(\frac{(x+y)^2 m_e^2}{(x+y)^2 m_e^2 + xy\mu^2}\right)\right\} \\ & \times\left\{ 1 + \frac{2}{2}\frac{e_r^2}{16\pi^2}\int_0^1 dx \ \ -2(1-x)\ln\left(\frac{x^2 m_e^2}{x m_e^2 + x(1-x)\mu^2}\right)\right\} \\ & \times\left\{ 1 + \frac{1}{2}\frac{e_r^2}{2\pi^2}\int_0^1 dx \ \ x(1-x)\ln\left(\frac{m_e^2 + x(1-x)\mu^2}{m_e^2}\right)\right\} \end{aligned}}$$

for $\mu^2 \ll m_e^2$: $\dfrac{\partial\, e_r(\mu)}{\partial \ln\mu} \sim \dfrac{e_r^3}{\pi^2} \times \left(\dfrac{m_e^2}{\mu^2}\right)$ power suppressed. $\Rightarrow$ ignore.

$m_e^2 \ll \mu^2$: $\dfrac{\partial\, e_r(\mu)}{\partial \ln(\mu^2)} \simeq e_r\left(-\dfrac{e^2}{16\pi^2} + \dfrac{e^2}{16\pi^2} + \dfrac{1}{2}\dfrac{e^2}{2\pi^2}\times\underbrace{\dfrac{1}{6}}_{\int_0^1 dx\, x(1-x)}\right)$

$= \dfrac{e^3}{24\pi^2}$

renormalization group equation at 1-loop. $\Rightarrow$ $\boxed{\dfrac{\partial(4\pi/e^2)}{\partial \ln\mu} = -\dfrac{2}{3\pi}}$

$(m_e \ll \mu \ll m_\mu)$

$$\boxed{\left(\frac{4\pi}{e^2}\right)(\mu) = \frac{1}{\alpha(\mu)} \simeq (\text{const}) - \frac{2}{3\pi}\ln\left(\frac{\mu}{m_e}\right)}$$

running coupling constant.

(*) correct only for log-part.

RG equation :

relation among renormalized coupling constants
for renormalized conditions
at different energy scales.

$$\frac{\partial g(\mu)}{\partial \ln \mu} = \frac{\partial}{\partial \ln(\sqrt{-q^2})}\left[g^{ren}(q^2)\right]\Bigg|_{q^2=-\mu^2} + g(\mu) \times \frac{\partial}{\partial \ln \mu} \ln\left[\prod_i \left(\frac{Z_i}{Z_i^{(\mu)}}\right)^{-1/2}\right]$$

(irreducible amplitudes (amputated) scale as $\prod_i \left(\frac{Z_i}{Z_i^{(\mu)}}\right)^{-1/2}$ (like coefficients))

$$= -\frac{\partial}{\partial \ln \Lambda}\left[g(q^2;\Lambda)\right] + g(\mu)\frac{\partial}{\partial \ln \Lambda}\ln\left(\prod_i Z_i^{-1/2}\right)$$

$$\begin{cases} \Delta g^r \sim \ln\left(\frac{\Lambda^2}{q^2}\right) \Rightarrow g^{ren} \sim \ln\left(\frac{m^2}{q^2}\right) \\ Z^{(\mu)} \sim \ln\left(\frac{\Lambda^2}{m^2+\mu^2}\right), \quad Z \sim \ln\left(\frac{\Lambda^2}{m^2}\right) \end{cases}$$

$$\begin{cases} \gamma \equiv -\frac{\partial}{\partial \ln \mu}\left(\ln\sqrt{\frac{Z}{Z^{(\mu)}}}\right) \\ \frac{\partial g(\mu)}{\partial \ln \mu} \equiv \beta_g. \end{cases}$$

(β- fcn : determined from log divergence part.)

Dimensional Regularization

An easy way to $\left\{ \begin{array}{l} \text{calculate } \beta\text{-fcn} \\ \text{renormalize. (regularize \& subtract)} \end{array} \right\}$

Loop momentum integration.

$$\frac{d^4k}{(2\pi)^4} \Rightarrow \frac{d^nk}{(2\pi)^n}(\mu)^{4-n} \Rightarrow i\frac{vol(S_{n-1})}{(2\pi)^n \cdot 2}\int dK\, K^{\frac{n}{2}-1}$$

after appropriate shift.

$vol(S_{n-1})$:

$$\int d^nx\, e^{-\sum_i^n (x_i)^2} = \left(\int_{-\infty}^{+\infty} dx\, e^{-x^2}\right)^n = \pi^{\frac{n}{2}}$$

$$\parallel$$

$$\int_0^{+\infty} dr\, r^{n-1} vol(S_{n-1}) \cdot e^{-r^2} = \frac{vol(S_{n-1})}{2} \cdot \int_0^{+\infty} dR\, R^{\frac{n}{2}-1} e^{-R} = \frac{vol(S_{n-1})}{2}\Gamma\left(\frac{n}{2}\right)$$

$(R = r^2)$

$$\Rightarrow \boxed{\frac{vol(S_{n-1})}{2} = \frac{\pi^{n/2}}{\Gamma(n/2)}}$$

Idea: $$\int_0^{\Lambda^2} dK \frac{K}{[K+m^2-x(1-x)q^2]^2} \simeq \ln\left(\frac{\Lambda^2+m^2-x(1-x)q^2}{m^2-x(1-x)q^2}\right) - 1.$$

Logarithmically divergent.

but
$$\lim_{\Lambda\to+\infty}\int^{\Lambda^2} dK \frac{K^{\frac{n}{2}-1}\mu^{4-n}}{[K+m^2-x(1-x)q^2]^2} = \left(\frac{\mu^2}{m^2-x(1-x)q^2}\right)^{2-\frac{n}{2}}\int_0^{+\infty} dy\, \frac{y^{\frac{n}{2}-1}}{(y+1)^2}$$

$$\parallel$$

$$\int_0^1 d\xi\, (1-\xi)^{\frac{n}{2}-1}\xi^{1-\frac{n}{2}}$$

convergent if $\frac{n}{2} > 0$ and $2 - \frac{n}{2} > 0 \Leftrightarrow \boxed{4 > n}$

$$= \left(\frac{\mu^2}{m^2-x(1-x)q^2}\right)^{2-\frac{n}{2}}\frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(2-\frac{n}{2}\right)}{\Gamma(2)}$$

$$\int_0^1 dx \int \frac{d^n k}{(2\pi)^n} \mu^{4-n} \frac{1}{[k^2 - m^2 + x(1-x)q^2]^2} = \underset{\int_0^1 dx}{\wedge}\, i \frac{\pi^{n/2}}{(2\pi)^n \Gamma(n/2)} \Gamma\left(\frac{n}{2}\right) \Gamma\left(2-\frac{n}{2}\right) \left(\frac{\mu^2}{m^2 - x(1-x)q^2}\right)^{2-\frac{n}{2}}$$

$$= \int_0^1 dx \frac{i}{(4\pi)^2} \underbrace{\Gamma\left(2-\frac{n}{2}\right)\left(\frac{4\pi\mu^2}{m^2 - x(1-x)q^2}\right)^{2-\frac{n}{2}}}$$

small $\left(2-\frac{n}{2}\right)$ $\quad$ $(-\gamma = -0.5772)$

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} = \frac{1}{z}\left(\Gamma(1) + \left.\frac{d\Gamma}{dz}\right|_{z=1} \times z + \cdots\right)$$

$$\left[\frac{1}{\left(2-\frac{n}{2}\right)} + (-\gamma) + \cdots\right]\left[1 + \left(2-\frac{n}{2}\right) \ln\left(\frac{\mu^2 4\pi}{m^2 - x(1-x)q^2}\right) + \cdots \right]$$

$$= \frac{1}{\left(2-\frac{n}{2}\right)} + (-\gamma) + \ln\left(\frac{\mu^2 4\pi}{m^2 - x(1-x)q^2}\right) + O\left(2-\frac{n}{2}\right).$$

- still divergent. when $n \to 4$.
- empirical rule: $\dfrac{1}{\left(2-\frac{n}{2}\right)} \Longleftrightarrow \ln(\Lambda^2)$

$\left(\nexists \ \dfrac{1}{(1-n/2)} \text{ pole} \Leftrightarrow \nexists \text{ quadratic divergence.}\right)$

β-function as coefficients of $\ln(\Lambda^2)$

$\Rightarrow \dfrac{1}{(2-n/2)}$;

- renormalization at scale μ.

simply subtract $\dfrac{1}{\left(2-\frac{n}{2}\right)} + (-\gamma + \ln(4\pi))$

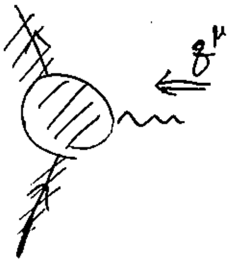
renormalization scheme. : ~~mini~~ $\overline{MS}$

(minimal subtraction or)

§ 6.3 Meaning of Running Coupling Constants. I

☆ Observables (eg. $|\mathcal{M}|^2$ for a given kinematics)

should not depend on the choice of renormalization scale.



=

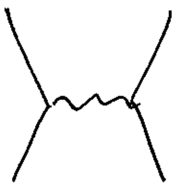
+ radiative correction.

$\approx \left(i e_r(|q|) \gamma^\mu \right)$ $\quad = i e_r \gamma^\mu$

$\Uparrow$ due to the difference between $\vec{q} = \vec{0}$ & $\vec{q} \neq \vec{0}$.

☆ good approximation <u>at fixed order perturbation</u>.

eg. QED scattering amplitude.

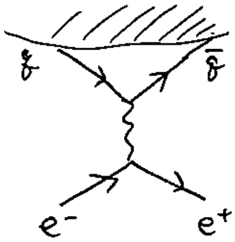


+ ...

if at tree level. $\left(i\mathcal{M} \sim i \frac{e_r^2(\mu)}{q^2} \eta_{\mu\nu} \times (\text{polarizatio}\right.$

corrections of order $\times \frac{\alpha_e}{\pi} \times \ln\left(\frac{-q^2}{\mu^2}\right)$ remain.

eg. total hadron σ



$$\sigma_{tot} = \frac{4\pi\alpha_e^2}{3s}(Q_q)^2 \times 3 \times \left[1 + \frac{\alpha_s(\mu^2)}{\pi} + \left(\frac{\alpha_s(\mu^2)}{\pi}\right)^2 \left\{ c_2 + \pi b \ln\left(\frac{s}{\mu^2}\right) \right\} + \left(\frac{\alpha_s(\mu^2)}{\pi}\right)^3 \left[c_3 + \left\{\pi b \ln\left(\frac{s}{\mu^2}\right)\right\}^2 - \cdots \ln\left(\frac{s}{\mu^2}\right) \right] + \cdots \right]$$

$$\left(\frac{\partial}{\partial \ln\mu^2} \left(\frac{1}{\alpha_s(\mu^2)} \right) = b + \mathcal{O}(\alpha_s). \right)$$

$\boxed{\text{take } \mu^2 \simeq s\ !}$

☆ resum $\sum_k \left(\frac{\alpha_s}{\pi}\right)^k \left(\ln\left(\frac{\mu_1^2}{\mu_0^2}\right)\right)^k$ $\qquad \alpha_s(\mu_1) \cong \frac{\alpha_s(\mu_0)}{1 + \alpha_s(\mu_0)\, b \ln\left(\frac{\mu_1^2}{\mu_0^2}\right)}$

↑ leading log resummation.