RG equation :

relation among renormalized coupling constants
for renormalized conditions
at different energy scales.

$$\frac{\partial g(\mu)}{\partial \ln \mu} = \frac{\partial}{\partial \ln(\sqrt{-q^2})}\left[g^{ren}(q^2)\right]\Bigg|_{q^2=-\mu^2} + g(\mu) \times \frac{\partial}{\partial \ln \mu} \ln\left[\prod_i \left(\frac{Z_i}{Z_i^{(\mu)}}\right)^{-1/2}\right]$$

(irreducible amplitudes (amputated) scale as $\prod_i \left(\frac{Z_i}{Z_i^{(\mu)}}\right)^{-1/2}$ (like coefficients))

$$= -\frac{\partial}{\partial \ln \Lambda}\left[g(q^2;\Lambda)\right] + g(\mu)\frac{\partial}{\partial \ln \Lambda}\ln\left(\prod_i Z_i^{-1/2}\right)$$

$$\begin{cases} \Delta g^{r} \sim \ln\left(\frac{\Lambda^2}{q^2}\right) \Rightarrow g^{ren} \sim \ln\left(\frac{m^2}{q^2}\right). \\ Z^{(\mu)} \sim \ln\left(\frac{\Lambda^2}{m^2+\mu^2}\right), \quad Z \sim \ln\left(\frac{\Lambda^2}{m^2}\right) \end{cases}$$

$$\begin{cases} \gamma \equiv -\frac{\partial}{\partial \ln \mu}\left(\ln\sqrt{\frac{Z}{Z^{(\mu)}}}\right) \\ \frac{\partial g(\mu)}{\partial \ln \mu} \equiv \beta_g. \end{cases}$$

(β- fcn : determined from log divergence part.)

Dimensional Regularization

An easy way to $\begin{cases} \text{calculate } \beta\text{-fcn} \\ \text{renormalize. (regularize \& subtract)} \end{cases}$

Loop momentum integration.

$$\frac{d^4k}{(2\pi)^4} \Rightarrow \frac{d^nk}{(2\pi)^n}(\mu)^{4-n} \Rightarrow i\frac{vol(S_{n-1})}{(2\pi)^n\cdot 2}\int dK\, K^{\frac{n}{2}-1}$$

after appropriate shift.

$vol(S_{n-1})$:

$$\int d^nx\, e^{-\sum_i (x_i)^2} = \left(\int_{-\infty}^{+\infty} dx\, e^{-x^2}\right)^n = \pi^{\frac{n}{2}}$$

$$\parallel$$

$$\int_0^{+\infty} dr\, r^{n-1}\, vol(S_{n-1})\cdot e^{-r^2} = \frac{vol(S_{n-1})}{2}\cdot\int_0^{+\infty} dR\, R^{\frac{n}{2}-1}e^{-R} = \frac{vol(S_{n-1})}{2}\Gamma\left(\frac{n}{2}\right)$$

$(R=r^2)$

$$\Rightarrow \boxed{\frac{vol(S_{n-1})}{2} = \frac{\pi^{n/2}}{\Gamma(n/2)}}$$

Idea: $$\int_0^{\Lambda^2} dK\frac{K}{[K+m^2-x(1-x)q^2]^2} \cong \ln\left(\frac{\Lambda^2+m^2-x(1-x)q^2}{m^2-x(1-x)q^2}\right) - 1.$$

Logarithmically divergent.

but
$$\lim_{\Lambda\to+\infty}\int^{\Lambda^2} dK\frac{K^{\frac{n}{2}-1}\,\mu^{4-n}}{[K+m^2-x(1-x)q^2]^2} = \left(\frac{\mu^2}{m^2-x(1-x)q^2}\right)^{2-\frac{n}{2}}\int_0^{+\infty}dy\,\frac{y^{\frac{n}{2}-1}}{(y+1)^2}$$

$$\parallel$$

$$\int_0^1 d\xi\,(1-\xi)^{\frac{n}{2}-1}\xi^{1-\frac{n}{2}}$$

convergent if $\frac{n}{2}>0$ and $2-\frac{n}{2}>0 \Leftrightarrow \boxed{4>n}$

$$= \left(\frac{\mu^2}{m^2-x(1-x)q^2}\right)^{2-\frac{n}{2}}\frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(2-\frac{n}{2}\right)}{\Gamma(2)}$$

$$\int_0^1 dx \int \frac{d^n k}{(2\pi)^n} \mu^{4-n} \frac{1}{[k^2 - m^2 + x(1-x)q^2]^2} = \int_0^1 dx\; i \frac{\pi^{n/2}}{(2\pi)^n \Gamma(n/2)} \Gamma\left(\frac{n}{2}\right) \Gamma\left(2-\frac{n}{2}\right) \left(\frac{\mu^2}{m^2 - x(1-x)q^2}\right)^{2-\frac{n}{2}}$$

$$= \int_0^1 dx \frac{i}{(4\pi)^2} \underbrace{\Gamma\left(2-\frac{n}{2}\right) \left(\frac{4\pi\mu^2}{m^2 - x(1-x)q^2}\right)^{2-\frac{n}{2}}}$$

small $\left(2-\frac{n}{2}\right)$ $(-\gamma = -0.5772)$

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} = \frac{1}{z}\left(\Gamma(1) + \left.\frac{d\Gamma}{dz}\right|_{z=1} \times z + \cdots\right)$$

$$\left[\frac{1}{(2-\frac{n}{2})} + (-\gamma) + \cdots\right]\left[1 + \left(2-\frac{n}{2}\right) \ln\left(\frac{\mu^2 4\pi}{m^2 - x(1-x)q^2}\right) + \cdots\right]$$

$$= \frac{1}{(2-\frac{n}{2})} + (-\gamma) + \ln\left(\frac{\mu^2 4\pi}{m^2 - x(1-x)q^2}\right) + O\left(2-\frac{n}{2}\right).$$

- still divergent. when $n \to 4$.
- empirical rule: $\frac{1}{(2-\frac{n}{2})} \Longleftrightarrow \ln(\Lambda^2)$

$\left(\exists\ \frac{1}{(1-n/2)} \text{ pole} \Leftrightarrow \exists \text{ quadratic divergence.}\right)$

β-function as coefficients of $\underline{\ln(\Lambda^2)}$

$\Rightarrow \underline{\frac{1}{(2-n/2)}}$

- renormalization at scale μ.

simply subtract $\frac{1}{(2-\frac{n}{2})} + \left(-\gamma + \ln(4\pi)\right)$

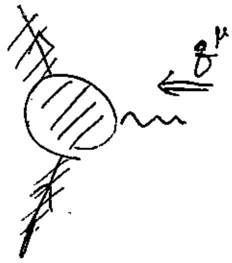
renormalization scheme. : ~~mini~~ $\overline{MS}$

(minimal subtraction or)

§ 6.3 Meaning of Running Coupling Constants. I

☆ Observables (eg. $|\mathcal{M}|^2$ for a given kinematics)

should not depend on the choice of renormalization scale.



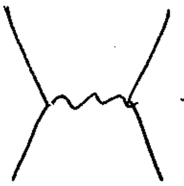
=

\+ radiative correction.

$\simeq \left(i e_r(|q|)\gamma^\mu\right)$ $\quad =i e_r \gamma^\mu$ $\quad \Uparrow$ due to the difference between $\vec{q}=\vec{0}$ & $\vec{q}\neq\vec{0}$.

☆ good approximation at fixed order perturbation.

eg. QED scattering amplitude.

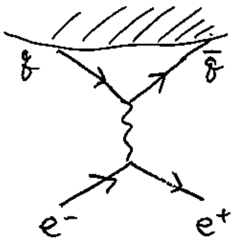


\+ ...

if at tree level. $\left(i\mathcal{M} \sim i \frac{e_r^2(\mu)}{q^2}\eta_{\mu\nu}\times(\text{polarizati}\right.$

corrections of order $\times \frac{\alpha_e}{\pi} \times \ln\left(\frac{-q^2}{\mu^2}\right)$ remain.

eg. total hadron σ



$$\sigma_{tot} = \frac{4\pi\alpha_e^2}{3s}(Q_q)^2 \times 3 \times \left[1 + \frac{\alpha_s(\mu^2)}{\pi} + \left(\frac{\alpha_s(\mu^2)}{\pi}\right)^2 \left\{ C_2 + \pi b \ln\left(\frac{s}{\mu^2}\right)\right\} + \left(\frac{\alpha_s(\mu^2)}{\pi}\right)^3 \left[C_3 + \left\{\pi b \ln\left(\frac{s}{\mu^2}\right)\right\}^2 - \cdots \ln\left(\frac{s}{\mu^2}\right) \right] + \cdots \right]$$

$$\left(\frac{\partial}{\partial \ln\mu^2}\left(\frac{1}{\alpha_s(\mu^2)}\right) = b + \mathcal{O}(\alpha_s^{*}). \right)$$

$\boxed{\text{take } \mu^2 \simeq s \text{ !}}$

☆ resum $\displaystyle \sum_k \left(\frac{\alpha_s}{\pi}\right)^k \left(\ln\left(\frac{\mu_1^2}{\mu_0^2}\right)\right)^k$ $\qquad \displaystyle \alpha_s(\mu_1) \simeq \frac{\alpha_s(\mu_0)}{1+\alpha_s(\mu_0)\, b \ln\left(\frac{\mu_1^2}{\mu_0^2}\right)}$

$\nearrow$ leading log resummation.

§6.4 Wilson's interpretation of renormalization group.

(meaning of running coupling constants II)

technically ....

eg. quantum correction. $\Pi^{(1)}_{\mathrm{ren.}(\mu)}(q^2)$

$$= \frac{e^2}{2\pi^2}\int_0^1 dx\; x(1-x)\left\{\ln\left(\frac{m_e^2 - x(1-x)q^2}{M_{\mathrm{reg}}^2}\right) - \underbrace{\ln\left(\frac{m_e^2 + x(1-x)\mu^2}{M_{\mathrm{reg}}^2}\right)}_{\Uparrow\ \text{counter term.}}\right\}$$

or... in fermion wavefcn renormalization. (subtract)

Originally ... from integrals like.

$$\int\frac{d^4k}{(2\pi)^4}\frac{1}{[k^2 + m^2 + x(1-x)q^2]^2} \Rightarrow i\frac{1}{16\pi^2}\int_0^{\Lambda^2} dK\,\frac{K}{[K + m^2 - x(1-x)q^2]^2}$$

$$\approx \int^{\Lambda^2}\frac{dK}{K} \Longrightarrow \int^{\mu^2}\frac{dK}{K}$$

at large $K = (k^2)_E \gg m^2, |q^2|$ etc.

$\boxed{\text{effectively replaced in this way}}$

If we take $\mu^2 \simeq |q^2|$ (kinematics of interest)

Theory: $\boxed{\text{regularized at } \Lambda^2, M_{\mathrm{reg}}^2 \text{ with } M, e.}$ $\Longleftrightarrow$ Theory. $\boxed{\text{Loop momentum integration cut-off at } \mu^2. \text{ with } m(\mu)\ e_r(\mu)}$

In path-integral language.

$$\boxed{Z = \int_{|k|\lesssim\Lambda, M_{\mathrm{reg}}} \mathcal{D}A_\mu\mathcal{D}\Psi\mathcal{D}\bar\Psi\, e^{iS[A,\Psi,\bar\Psi;\,M,e]}} \Longrightarrow \boxed{Z = \int_{|k|\lesssim\mu} \mathcal{D}A_\mu\mathcal{D}\Psi\mathcal{D}\bar\Psi \left(e^{iS[\ \ ;\,m(\mu),e_r(\mu)]} + \cdots\right.}$$

$$S \to S + \left(\int A_\mu J^\mu + \int \Psi^\dagger K + h.c.\right)$$

$Z[J, K]$.

If interested only in $Z[J,K]$ for $J(q)$, $K(p)$ etc. integrate $A(k)$, $\Psi(k)$ etc. $|q|, |p| \ll |k|$

Fine. 1

(+ --) part. → (add non-renormalizable operators)

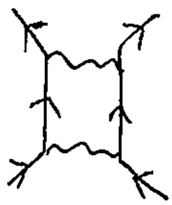
low mom. + high mom

⇒

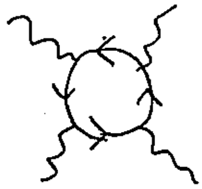
low mom. only ($|k| \le \mu$.)

and $A_\mu \cdot \Psi \cdot e_r(\mu)$ different normalizatio

⇒ already seen.



(low + high) loop momentum ⇒ low mom. only



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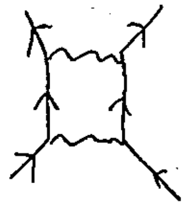

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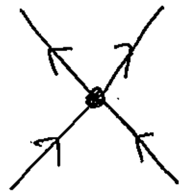


extra terms. $+\frac{1}{M^\alpha} F..F..F..F..$, $+(\bar\Psi\Psi\bar\Psi\Psi)\frac{1}{M^2}$

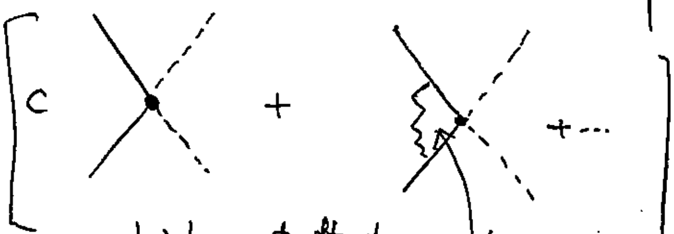
- $\int dK \frac{K}{[K+m^2-q^2]^\alpha}$ [$\int dK$ for $\mu^2 < K$].
- such integrals ... dominated by IR region.

→ unimportant if $m^2, |q|^2 \ll \mu^2$.

Non-renormalizable operators

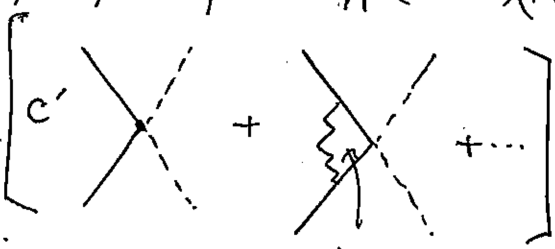
like { dim.5 ν mass $\Delta\mathcal{L} = (\bar\Psi\Psi\phi\phi)\frac{1}{M}$.

β-decay 4-fermi op. $\Delta\mathcal{L} = \frac{1}{M^2}(\bar\Psi\gamma^\mu\Psi)(\bar\Psi\gamma_\mu$



high cut-off theory. low / high } loop momentum.

⇒



low mom. only $|k| < \mu$.

C′: μ-dependent.

§ 7. Low-energy Effective Theory

Wilson's interpretation on "renormalization at scale μ"

When all the external lines (probes) have momenta $\ll \mu$.

integrate out all the D.O.F. w/. $k \gtrsim \mu$ first!
(take care of)

$\Longrightarrow$ change coupling constant values. (renormalize coupling constants

additional operators.

(effectively do the job of high mom. mode

Take one step further.

✓ for a massive particle ϕ with $\mu \ll M_\phi$.

some $\mathcal{D}\phi(k)$ left. but no on-shell modes for $k^\mu \ll \mu$.

$\longrightarrow$ does not come out as an external state.

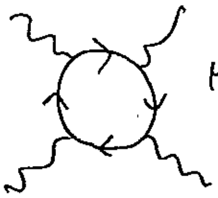
$\Longrightarrow$ integrate out ϕ completely! get it done!

e.g. 1. QED w/ $e^+ . e^-$, $\mu^+ . \mu^-$

$m_e \simeq 0.511$ MeV, $m_\mu \simeq 105$ MeV.

what if we take $\boxed{m_e \ll \mu \ll m_\mu}$??

(i)



$\mu^\pm$ loop.

carry out this integral completely.

$$\Rightarrow \Delta\mathcal{L} = \frac{2\alpha_{QED}^2}{45 m_\mu^4}\left\{\frac{1}{4}(F_{\mu\nu}F^{\mu\nu})^2 + \frac{7}{16}\left(F_{\mu\nu}F_{\kappa\lambda}\frac{\epsilon^{\mu\nu\kappa\lambda}}{2}\right)^2\right\}$$

(ii) photon propagator. $\frac{e^2}{q^2(1-\Pi_{ren})}$

$$\boxed{\left(\frac{1-\Pi_{ren}(q^2)}{e^2}\right) = \frac{1}{e^2_{*(e\mu)}(\cancel{\mu}E)} - \frac{1}{2\pi^2}\int_0^1 dx\, x(1-x)\sum_i \ln\left(\frac{m_i^2 - x(1-x)q^2}{m_i^2 + x(1-x)E^2}\right)}$$

$$\left(\text{cf.} \quad = \frac{1}{e^2_{\overline{MS}}(E)} - \frac{1}{2\pi^2}\int_0^1 dx\, x(1-x)\sum_i \ln\left(\frac{m_i^2 - x(1-x)q^2}{E^2}\right)\right).$$

scheme dependence.

☆ $$\frac{1}{e^2_{*(e\mu)}(E)} = \frac{1}{e^2_{*(e\mu)}(E_0)} - \frac{1}{2\pi^2}\int_0^1 dx\, x(1-x)\sum_i \ln\left(\frac{m_i^2 + x(1-x)E^2}{m_i^2 + x(1-x)E_0^2}\right)$$

$$\boxed{\frac{\partial(1/e^2_{*(e\mu)})}{\partial \ln(E^2)} = -\frac{1}{2\pi^2}\sum_i\int_0^1 dx\, x(1-x)\left(\frac{x(1-x)E^2}{m_i^2 + x(1-x)E^2}\right)} \to \begin{cases} 1/6 & (m_i^2 \ll \ldots \\ \frac{1}{30}\left(\frac{E^2}{m_i^2}\right) & (E^2 \ll \ldots \end{cases}$$

$\ll 1$.

☆ when $E \ll m_\mu$.

$$\frac{1}{e^2_{*(e)}(E)} \equiv \frac{1}{e^2_{*(e\mu)}(E)} - \frac{1}{2\pi^2}\int dx\, x(1-x)\ln\left(\frac{m_\mu^2}{m_\mu^2 + x(1-x)E^2}\right)$$

$$\Rightarrow \left(\frac{1-\Pi_{ren}(q^2)}{e^2}\right)\cdot q^2 = q^2\left\{\frac{1}{e^2_{*(e)}(E)} - \frac{1}{2\pi^2}\int_0^1 dx\, x(1-x)\ln\left(\frac{m_e^2 - x(1-x)q^2}{m_e^2 + x(1-x)E^2}\right)\right\} + \frac{1}{2\pi^2}\left(\frac{1}{30}\right)\frac{(q^2)^2}{m_\mu^2} + \cdots$$

$\Downarrow$

quantum correction in QED w/ $e^+ . e^-$ only.

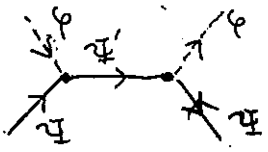
$\Downarrow$

effective operator

$\Delta\mathcal{L} \propto \left(\frac{e^2}{2\pi^2}\right)\frac{1}{30}\frac{1}{m_\mu^2}(F_{\mu\nu}\partial^2$

eg. 2 "see-saw mechanism" $\Leftarrow$ [Weyl fermion (not Dirac fermion) should be used in reality]

$$\mathcal{L} = \bar{\Psi}(i\gamma\cdot D)\Psi + \bar{\Psi}'(i\gamma\cdot D)\Psi' - M\bar{\Psi}'\Psi'$$
$$+ (D^\mu\varphi)^*(D_\mu\varphi) \qquad + \lambda\bar{\Psi}'\Psi\varphi + \lambda^*\bar{\Psi}\Psi'\varphi^*$$



$\boxed{\text{at energy scale } E \ll M.}$

(incl. momentum transfer)

propagator $\frac{i(\not{q}+M)}{q^2-M^2+i\epsilon} \simeq -i\frac{M}{M^2} = \frac{-i}{M}$.

$$\cancel{\mathcal{L}_{eff}} = i\mathcal{M} \sim (i\lambda\cdot i\lambda^*)[\varphi^*\bar{\Psi}]\left(\frac{-i}{M}\right)[\varphi\Psi]$$

$\Uparrow$ reproduce.

$$\boxed{\mathcal{L}_{eff} = \frac{|\lambda|^2}{M}(\varphi^*\bar{\Psi}\Psi\varphi)}$$

in theory w/o Ψ'

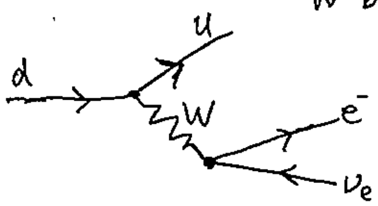
$\langle\varphi\varphi^*\rangle \sim v^2$

$\Rightarrow \Psi_{mass} \sim \frac{|\lambda|^2 v^2}{M}$

tiny if $v \ll M$.

eg. 3. 4-fermi operator. (β-decay)

at $E \ll m_W$ W-boson propagator $\frac{-i\eta_{\mu\nu}}{q^2-m_W^2} \Rightarrow \frac{i\eta_{\mu\nu}}{m_W^2}$



$$\boxed{\mathcal{L}_{eff.} = \left[\bar{u}\gamma^\mu\left(\frac{1-\gamma_5}{2}\right)d\right]\left[\bar{e}\gamma_\mu\left(\frac{1-\gamma_5}{2}\right)\nu\right]\left(\frac{-g^2}{2m_W^2}\right)}$$

$\uparrow$ in QCD×QED

$$i\mathcal{M} = (-ig)\left[\bar{e}\gamma^\mu\left(\frac{1-\gamma_5}{2}\right)\nu\right]\frac{-i\eta_{\mu\nu}}{q^2-m_W^2}(-ig)\left[\bar{u}\gamma^\nu\left(\frac{1-\gamma_5}{2}\right)d\right]\frac{1}{2}$$
$$\hookrightarrow \simeq -i\frac{g^2}{2m_W^2}[\bar{e}\cdots\nu]^\mu[\bar{u}\cdots d]_\mu.$$

$\boxed{\text{Low-Energy Effective Theory}}$

heavy particles can be integrated out for low energy description.

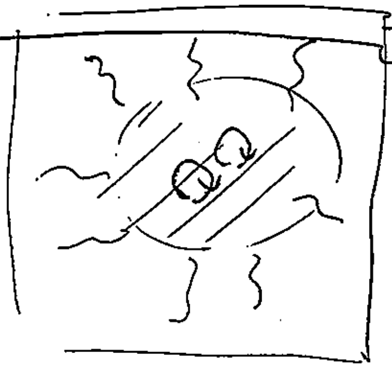
{ couplings : renormalized.
effective operators generated.. no other footprints.

Standard Model is yet another effective theory of some more fund. theory

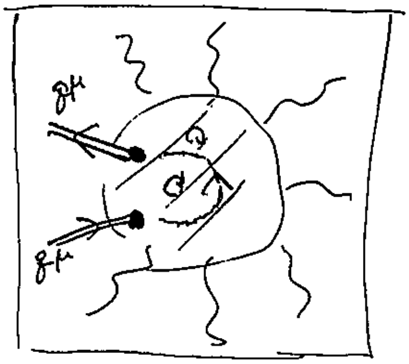
§8. Operator Product Expansion.

Low-energy ~~Eff.~~ Theory

by carrying out high-mom. loop first.



[all ext. mom. are soft.]



[all ext. mom. are soft except 2]

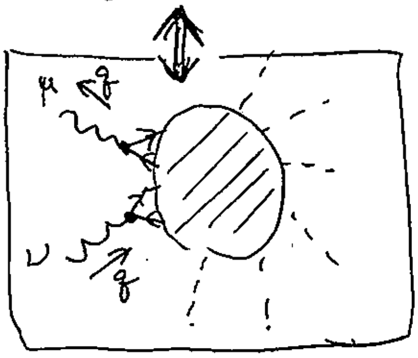
Two operators combined ⇒ no net momentum. flow outside.

Any effective description.

Consider.

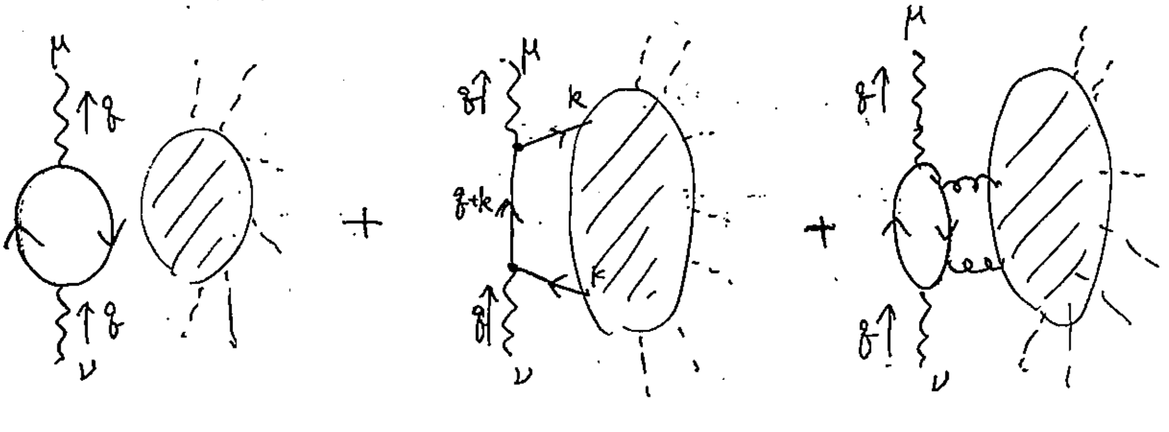
$$-e^2 \iint \langle \Omega | T\{ \cdots J^\mu(x)\, J^\nu(y) \cdots \} | \Omega \rangle\, e^{iq'\cdot x}\, e^{-iq\cdot y}\, d^4x\, d^4y .$$

$\Updownarrow$



large momentum q^μ. $(q \simeq q')$

necessarily flow from $J^\nu(y)$ to $J^\mu(x)$.



$$\boxed{\int [ie J^\mu(x)]\,[ie J^\nu(y)]\, e^{iq'\cdot x}\, e^{-iq\cdot y}\, d^4y \quad \text{in } T\{\cdots\}}$$

$$= i\left(q^2\eta^{\mu\nu} - q^\mu q^\nu\right)\Pi^{(1)}_{\text{ren}}(q^2)\cdot e^{i(q'-q)\cdot x}\ \mathbb{1}.$$

$$+ \int d^4y\ e^{iq'\cdot x}\,[ie\bar\psi(x)\gamma^\mu]\ \underbrace{\int\frac{d^4k}{(2\pi)^4}\frac{i[(\not q+\not k)+m]\,e^{-i(q+k)\cdot(x-y)}}{(q+k)^2-m^2+i\varepsilon}}_{\parallel}\ (ie)\,[\gamma^\nu\psi(y)]\,e^{-iq\cdot y}$$

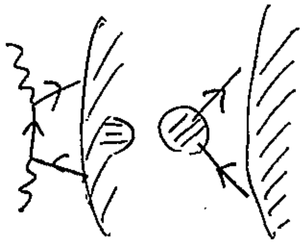
$\langle 0|T\{\psi_I(x)\bar\psi_I(y)\}|0\rangle$. propagator.

approximation. (or expansion).

$$\frac{i[(\not q+\not k)+m]}{(q+k)^2-m^2+i\varepsilon} \longrightarrow \frac{i\not q}{q^2}$$

$$\int d^4y\,(-ie^2)\frac{q_\lambda}{q^2}\int\frac{d^4k}{(2\pi)^4}\, e^{i(q'-q-k)\cdot x}\, e^{i(q+k-q)\cdot y}\,[\bar\psi(x)\gamma^\mu\gamma^\lambda\gamma^\nu\psi(y)]$$

$$\left(\int\frac{d^4k}{(2\pi)^4}\,e^{-ik\cdot(x-y)} = \delta^4(x-y)\right)$$



$$= -ie^2\frac{q_\lambda}{q^2}\,e^{i(q'-q)\cdot x}\,[\bar\psi(x)\gamma^\mu\gamma^\lambda\gamma^\nu\psi(y)]$$

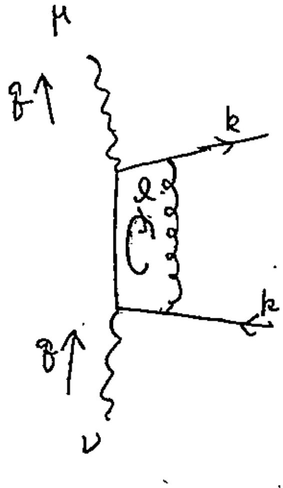
$$= \sum_I C_I(q^\kappa)\cdot \mathcal{O}_I(x).$$

k^μ's : momentum in $\psi(y)$ or $\bar\psi(x)$.

$\Rightarrow \partial$ on ψ or ∂ on $\bar\psi$.

$\Rightarrow$ derivative expansion. $\dfrac{q\cdot\overleftrightarrow{\partial}}{q^2}\,\mathcal{O}$.

loop correction.



$|q| \ll \mu$: finite integral (ℓ UV)

$\mu \ll q$: log ~~divergence~~ correction. (ℓ)

propagator $\left\{\dfrac{i(\not q+\not\ell+\not k)+m}{(q+\ell+k)^2-m^2} \simeq \dfrac{i\not q}{q^2}\right.$

$$C_I(q^2;\alpha_s;\mu)\ \ [\mathcal{O}_I(x)]_\mu.$$

$\{$ tree + 1-loop $\mu<\ell$ $\}$

only $k<\mu$.