No.

Date

§6.4 Wilson's interpretation of renormalization group

(meaning of running coupling constants II)

Remember that the procedure of renormalization is technically ...

eg. quantum correction $\Pi^{(1)}_{\mathrm{ren}(\mu)}(q^2)$

counter term

$$= \frac{(eQ)^2}{2\pi^2}\int_0^1 dx\, x(1-x)\left\{\ln\left(\frac{m_e^2 - x(1-x)q^2}{M_{\mathrm{reg}}^2}\right) - \ln\left(\frac{m_e^2 + x(1-x)\mu^2}{M_{\mathrm{reg}}^2}\right)\right\}$$

to render a divergent (large log) integral finite by removing the near-M_{reg} contribution.

$$\int\frac{d^4k}{(2\pi)^4}\frac{1}{[k^2-m^2+x(1-x)q^2]^2} \simeq \frac{i}{16\pi^2}\int_0^{\Lambda^2}\frac{dK\,K}{[K+m^2-x(1-x)q^2]^2}$$

$$\approx \frac{i}{16\pi^2}\int_{m^2,-q^2}^{\Lambda^2}\frac{dK}{K} \Longrightarrow \frac{i}{16\pi^2}\left\{\int_{m^2,-q^2}^{\Lambda^2}\frac{dK}{K} - \int_{\mu^2}^{\Lambda^2}\frac{dK}{K}\right\}$$

When renormalization conditions at the energy scale μ are employed, degrees of freedom with $\mu^2 \lesssim K$ are virtually not integrated over.

Set $\mu^2 \gtrsim$ (kinematical invariants of interest).

Theory regularized at Λ^2, M_{reg}^2 with M, e	$\Longleftrightarrow$	Theory loop momentum integrated only $K \lesssim \mu^2$ with $m(\mu)$, $e_r(\mu)$ $(+\cdots)$

In path-integral language

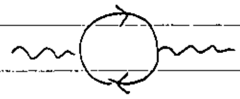
$$Z[J,K] = \int_{|k|\lesssim\Lambda,\,M_{\mathrm{reg}}}\mathcal{D}A_\mu\mathcal{D}\psi\; e^{iS[A,\psi;M,e]+i\int(A\cdot J+\psi\cdot K)} \Longrightarrow Z[J,K] = \int_{|k|<\mu}\mathcal{D}A\mathcal{D}\psi\; e^{iS_{(\mu)}+i\int(A\cdot J+K\cdot\psi)}$$

If interested only in $\langle A(p)\text{'s}\ \psi(p)\text{'s}\rangle$ with $p<\mu$
$Z[J(p), K(p)]$ with $p<\mu$ (low-energy physics)

OK. to integrate over $A(k)$, $\psi(k)$'s with $k>\mu$ first.

PLUS

$S_{(\mu)}[A, \psi, \cdots]$ cannot be the same as $S[A, \psi, M, e]$.

☆  $= i\left(q^2\eta_{\mu\nu} - q_\mu q_\nu\right)\underset{\substack{\text{full.}\\ \text{un-ren.}}}{\Pi(q^2)} =$  $+$ 

$(\mu \lesssim k)$ & $(k \lesssim \mu)$ $\qquad$ $(k \lesssim \mu)$ $\qquad$ $i\left(q^2\eta_{\mu\nu} - q_\mu q_\nu\right)\left(\Pi_{\mu\lesssim k}\right)$

short distance contrib.

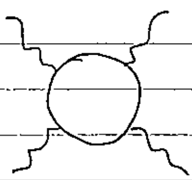
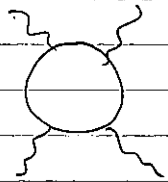
$$S_{(\mu)} \sim \int d^4x \; \underbrace{\frac{1}{2} A_\mu \partial^2 A^\mu}_{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}} - \frac{1}{2} A_\mu \partial^2 A^\mu \left(\Pi_{(\mu\lesssim k)}\right) + \cdots\cdots$$

$$=: \int d^4x \; \frac{1}{2}[A_\mu]_r \partial^2 [A^\mu]_r \qquad + \cdots\cdots$$

This definition of $[A_\mu]_r$ reproduces the relation

$$\underset{\text{renormalized field}}{[A_\mu]_r} = A_\mu \times \sqrt{(1 - \Pi_{\substack{\mu\lesssim k\\ \text{un-ren.}}})} \simeq A_\mu \cdot \sqrt{1 - \underset{\text{unren.}}{\Pi(\mu^2)}} = A_\mu \Big/ \sqrt{Z_3(\mu)}$$

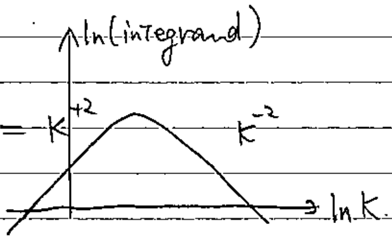
as expected.

☆  $\Longrightarrow$  $+$  $\Longleftarrow \Delta\mathcal{L}(\mu) = \frac{e^4}{(\mu^4)} F..F..F..F..$

$(\mu \lesssim k)$ & $(k \lesssim \mu)$ $\qquad$ $(k \lesssim \mu)$ $\qquad$ short distance contrib

In the end, though,

$$\int \frac{dK \quad K}{[K + m^2 - q^2]^4} =$$



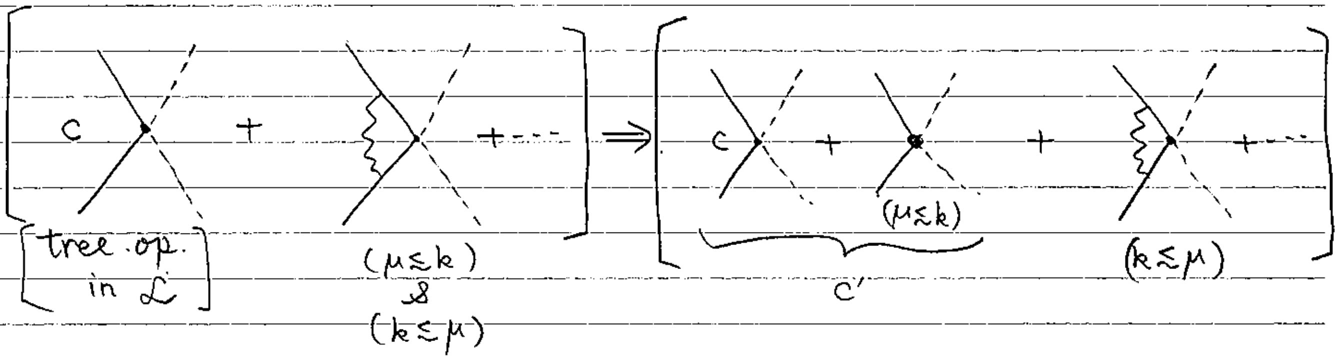
is dominated by the contribution from $K \sim (m^2 - q^2)$.

So, the effective operator $\Delta\mathcal{L}(\mu)$ is not important when $m^2 - q^2 \ll \mu^2$.

☆ non-renormalizable operator

(eg Majorana neutrino mass $\mathcal{L} = \frac{1}{M}(l\phi)(l\phi)$

The four-fermi op. $\mathcal{L} = G_F(\bar{\psi}\Gamma^{\mu}\psi)(\bar{\psi}\Gamma_{\mu}\psi)$)



$$\mathcal{L} \supset c(\psi\psi\phi\phi) \Rightarrow \mathcal{L}(\mu) \supset c'(\psi\psi\phi\phi) = \underbrace{c' Z_\psi^{(\mu)} Z_\phi^{(\mu)}}_{= c(\mu)} (\psi_r\psi_r\phi_r\phi_r)$$

<u>$c(\mu)$: logarithmic running.</u>

§ 7. Low-energy Effective Theory

Wilson's interpretation on "renormalization at scale μ."

When all the external lines (probes) have momenta $\ll \mu$.

integrate out all the D.O.F. w/. $k \gtrsim \mu$ first!
(take care of)

$\Longrightarrow$ change coupling constant values. (renormalize coupling constants

additional operators.

(effectively do the job of high mom. mode

Take one step further.

- for a massive particle ϕ with $\mu \ll M_\phi$.

  some $\mathcal{D}\phi(k)$ left. but no on-shell modes for $k^\mu \ll \mu$.

  $\longrightarrow$ does not come out as an external state.

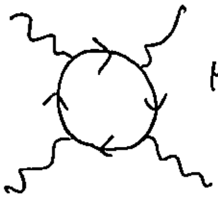
$\Longrightarrow$ integrate out ϕ completely! get it done!

eg. 1 QED w/ $e^+ e^-$, $\mu^+ \mu^-$

$$m_e \simeq 0.511 \text{ MeV}, \qquad m_\mu \simeq 105 \text{ MeV}.$$

what if we take $\boxed{m_e \ll \mu \ll m_\mu}$??

(i)



μ loop.

carry out this integral completely.

$$\Rightarrow \Delta\mathcal{L} = \frac{2\alpha_{QED}^2}{45 m_\mu^4}\left\{\frac{1}{4}\left(F_{\mu\nu}F^{\mu\nu}\right)^2 + \frac{7}{16}\left(F_{\mu\nu}F_{\kappa\lambda}\frac{\epsilon^{\mu\nu\kappa\lambda}}{2}\right)^2\right\}$$

(ii) photon propagator. $\dfrac{e^2}{q^2(1-\Pi_{ren})}$

$$\boxed{\left(\frac{1-\Pi_{ren}(q^2)}{e^2}\right) = \frac{1}{e^2_{*(e\mu)}(\cancel{4}E)} - \frac{1}{2\pi^2}\int_0^1 dx\, x(1-x)\sum_i \ln\left(\frac{m_i^2 - x(1-x)q^2}{m_i^2 + x(1-x)E^2}\right).}$$

$$\left(\text{c.f.} \quad = \frac{1}{e^2_{\overline{MS}}(E)} - \frac{1}{2\pi^2}\int_0^1 dx\, x(1-x)\sum_i \ln\left(\frac{m_i^2 - x(1-x)q^2}{E^2}\right)\right).$$

scheme dependence.

$$☆ \quad \frac{1}{e^2_{*(e\mu)}(E)} = \frac{1}{e^2_{*(e\mu)}(E_0)} - \frac{1}{2\pi^2}\int_0^1 dx\, x(1-x)\sum_i \ln\left(\frac{m_i^2 + x(1-x)E^2}{m_i^2 + x(1-x)E_0^2}\right)$$

$$\boxed{\frac{\partial\left(1/e^2_{*(e\mu)}\right)}{\partial \ln(E^2)} = -\frac{1}{2\pi^2}\sum_i\int_0^1 dx\; x(1-x)\left(\frac{x(1-x)E^2}{m_i^2 + x(1-x)E^2}\right)} \rightarrow \begin{cases} 1/6 & (m_i^2 \ll E^2) \\ \frac{1}{30}\left(\frac{E^2}{m_i^2}\right) & (E^2 \ll m_i^2) \end{cases}$$

$\ll 1.$

☆ when $E \ll m_\mu$.

$$\frac{1}{e^2_{*(e)}(E)} \equiv \frac{1}{e^2_{*(e\mu)}(E)} - \frac{1}{2\pi^2}\int dx\, x(1-x)\ln\left(\frac{m_\mu^2}{m_\mu^2 + x(1-x)E^2}\right)$$

$$\int_0^1 dx\, x^2(1-x)^2 = \frac{2^2}{5!}$$

$$\Rightarrow \left(\frac{1-\Pi_{ren}(q^2)}{e^2}\right)\cdot q^2 = q^2\left\{\frac{1}{e^2_{*(e)}(E)} - \frac{1}{2\pi^2}\int_0^1 dx\, x(1-x)\ln\left(\frac{m_e^2 - x(1-x)q^2}{m_e^2 + x(1-x)E^2}\right)\right\} + \frac{1}{2\pi^2}\left(\frac{1}{30}\right)\frac{(q^2)^2}{m_\mu^2} + \cdots$$

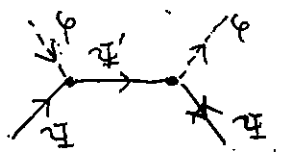
$\Downarrow$ quantum correction in QED w/ $e^+ e^-$ only.

$\Downarrow$ effective operator

$$\boxed{\Delta\mathcal{L} \propto \left(\frac{e^2}{2\pi^2}\right)\frac{1}{30}\frac{1}{m_\mu^2}\left(F_{\mu\nu}\partial^2 F^{\mu\nu}\right)}$$

eg. 2 "see-saw mechanism" $\Leftarrow$ [Weyl fermion (not Dirac fermion) should be used in reality]

$$\mathcal{L} = \bar{\Psi}(i\gamma\cdot D)\Psi + \bar{\Psi}'(i\gamma\cdot D)\Psi' - M\bar{\Psi}'\Psi' + (D^\mu\varphi)^*(D_\mu\varphi) + \lambda\bar{\Psi}'\Psi\varphi + \lambda^*\bar{\Psi}\Psi'\varphi^*$$



$\boxed{\text{at energy scale } E \ll M.}$ (incl. momentum transfer)

propagator $\frac{i(\not{q}+M)}{q^2-M^2+i\epsilon} \simeq -i\frac{M}{M^2} = \frac{-i}{M}.$

$$\cancel{\mathcal{L}_{eff}} = i\mathcal{M} \sim (i\lambda\cdot i\lambda^*)[\varphi^*\bar{\Psi}]\left(\frac{-i}{M}\right)[\varphi\Psi]$$

$\Uparrow$ reproduce.

$$\boxed{\mathcal{L}_{eff} = \frac{|\lambda|^2}{M}(\varphi^*\bar{\Psi}\Psi\varphi)}$$

in theory w/o Ψ'

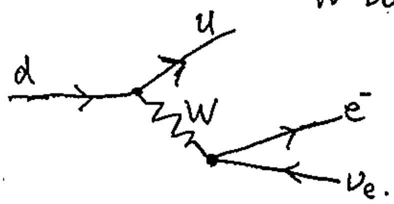
$\langle\varphi\varphi^*\rangle \sim v^2$

$\Rightarrow \Psi_{mass} \sim \frac{|\lambda|^2 v^2}{M}$

tiny if $v \ll M$.

eg. 3. 4-fermi operator. (β-decay)

at $E \ll m_W$ W-boson propagator $\frac{-i\eta_{\mu\nu}}{q^2-m_W^2} \Rightarrow \frac{i\eta_{\mu\nu}}{m_W^2}$



$$\boxed{\mathcal{L}_{eff.} = [\bar{u}\gamma^\mu\left(\tfrac{1-\gamma_5}{2}\right)d][\bar{e}\gamma_\mu\left(\tfrac{1-\gamma_5}{2}\right)\nu]\left(\frac{-g^2}{2m_W^2}\right)}$$

$\uparrow$ in QCD × QED

$$i\mathcal{M} = (-ig)[\bar{e}\gamma^\mu\left(\tfrac{1-\gamma_5}{2}\right)\nu]\frac{-i\eta_{\mu\nu}}{q^2-m_W^2}(-ig)[\bar{u}\gamma^\nu\left(\tfrac{1-\gamma_5}{2}\right)d]\frac{1}{2}$$

$$\hookrightarrow \simeq -i\frac{g^2}{2m_W^2}[\bar{e}\cdots\nu]^\mu[\bar{u}\cdots d]_\mu.$$

Low-Energy Effective Theory

heavy particles can be integrated out for low energy description.

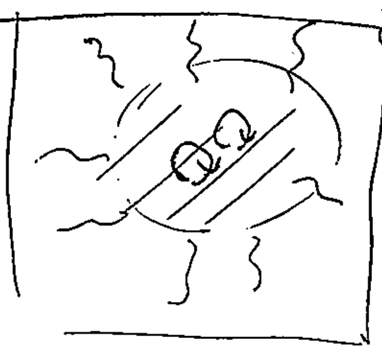
- couplings renormalized.
- effective operators generated.. no other footprints.

Standard Model is yet another effective theory of some more fund. theory

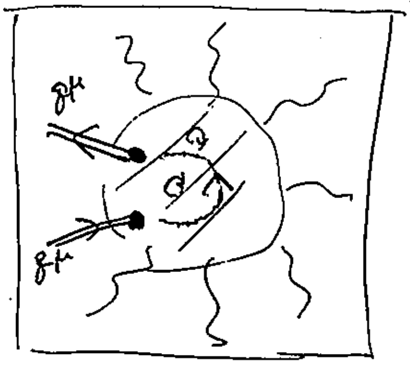
§8. Operator Product Expansion.

Low-energy Eff. Theory

by carrying out high-~~loop~~ mom. loop first.



[all ext. mom. are soft.]



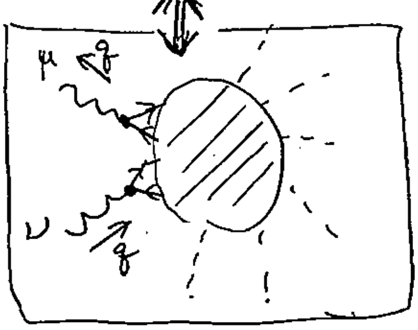
[all ext. mom. are soft except 2]

Two operators combined ⇒ no net momentum flow outside.

Any effective description.

Consider.

$$-e^2\iint \langle\Omega|T\{\cdots\ J^\mu(x)\, J^\nu(y)\ \cdots\}|\Omega\rangle\, e^{iq\cdot x}\, e^{-iq\cdot y}\, d^4x\, d^4y.$$

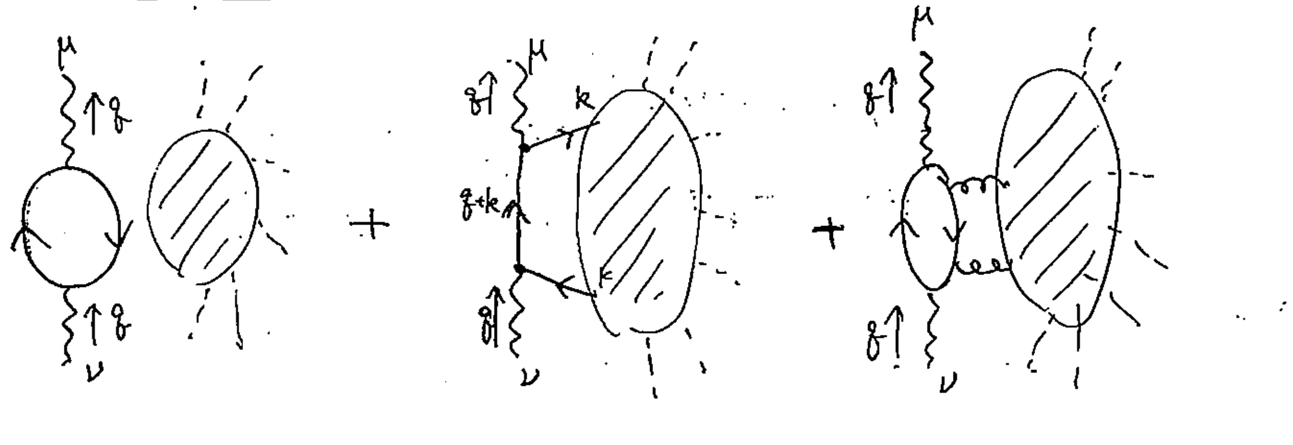


large momentum q^μ. ($q \simeq q'$)

necessarily flow from $J^\nu(y)$ to $J^\mu(x)$.

set μ so that (ext. mom.) $\ll \mu \ll q^2$.

ok. to integrate out $\mu \lesssim k$'s.



$$\boxed{\int [ie J^\mu(x)]\,[ie J^\nu(y)]\, e^{iq'\cdot x}\, e^{-iq\cdot y}\, d^4y \quad \text{in } T\{\cdots\}.}$$

$$= i\left(q^2\eta^{\mu\nu} - q^\mu q^\nu\right)\Pi^{(1)}_{\rm ren}(q^2)\cdot e^{i(q'-q)\cdot x}\ \mathbb{1}.$$

$$+ \int d^4y\ e^{iq'\cdot x}\left[ie\bar\psi(x)\gamma^\mu\right]\underbrace{\int\frac{d^4k}{(2\pi)^4}\frac{i[(\cancel{q}+\cancel{k})+m]}{(q+k)^2-m^2+i\varepsilon}e^{-i(q+k)\cdot(x-y)}}_{\parallel}\,(ie)\left[\gamma^\nu\psi(y)\right]e^{-iq\cdot y}$$

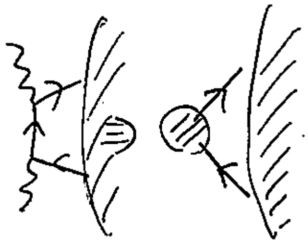
$\langle 0|T\{\psi_I(x)\bar\psi_I(y)\}|0\rangle$. propagator.

approximation. (or expansion).

$$\frac{i[(\cancel{q}+\cancel{k})+m]}{(q+k)^2-m^2+i\varepsilon} \to \frac{i\cancel{q}}{q^2}$$

$$\int d^4y\,(-ie^2)\frac{q_\lambda}{q^2}\int\frac{d^4k}{(2\pi)^4}e^{i(q'-q-k)\cdot x}\,e^{i(q+k-q)\cdot y}\left[\bar\psi(x)\gamma^\mu\gamma^\lambda\gamma^\nu\psi(y)\right]$$

$$\left(\int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot(x-y)} = \delta^4(x-y)\right)$$



$$\simeq\ -ie^2\frac{q_\lambda}{q^2}e^{i(q'-q)\cdot x}\left[\bar\psi(x)\gamma^\mu\gamma^\lambda\gamma^\nu\psi(y)\right]$$

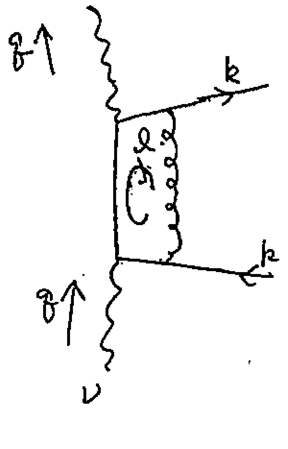
$$\boxed{= \sum_I C_I(q^\kappa)\cdot\mathcal{O}_I(x).}$$

k^μ's : momentum in $\psi(y)$ or $\bar\psi(x)$.

$\Rightarrow \partial$ on ψ or ∂ on $\bar\psi$.

$\Rightarrow$ derivative expansion. $\frac{q\cdot\overset{\to\text{or}\leftarrow}{\partial}}{q^2}\,\mathcal{O}$.

loop correction.



$|q| \ll \mu$: finite integral (ℓ UV)

$\mu \ll q$: log ~~divergence~~ correction. (ℓ)

propagator $\left\{\dfrac{i(\cancel{q}+\cancel{\ell}+\cancel{k})+m}{(q+\ell+k)^2-m^2} \simeq \dfrac{i\cancel{q}}{q^2}\right.$

$$\boxed{C_I(q^2;\alpha_s;\mu)\ [\mathcal{O}_I(x)]_\mu.}$$

$\{$ tree + 1-loop $\mu<\ell$ $\}$

only $k<\mu$.