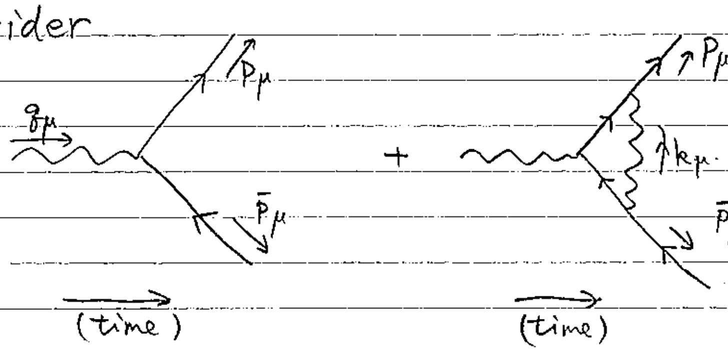
§9 Soft and Collinear Divergence

§9.1 Divergence in Virtual Corrections

Consider



$\gamma^* \to \bar{f}+f$ in QED.
(or $\gamma^* \to \bar{q}+q$ in QCD)

$$(-ieQ_f\Gamma^\mu) = -ieQ_f\gamma^\mu - ieQ_f\frac{(Q_fe)^2}{16\pi^2}\int dx\,dy\left[2\gamma^\mu\left\{\ln\left(\frac{(1-x-y)\Lambda^2+(x+y)^2m_f^2-xyq^2}{(x+y)^2m_f^2-xyq^2}\right) + \frac{m_f^2(1-4z+z^2)+(1-x)(1-y)q^2}{(x+y)^2m_f^2-xyq^2}\right\} + [\gamma^\mu,\gamma^\nu]\text{ term}\right]$$

at 1-loop.

(UV divergence manifest in $\ln(\Lambda^2)$ is gone when $\ln(\Lambda^2)\to\ln(\mu^2)$)

$z := (1-x-y)$

We still need to carry out $dx\,dy$ integration.

☆ fixed y: integral over small x region.

$$(\text{1-loop }\gamma^\mu) \simeq -2ieQ_f\frac{(Q_fe)^2}{16\pi^2}\,dy\int_0^{x_*}dx\,\frac{(1-y)q^2+m_f^2(1-4\cdots)}{y^2m_f^2+2xym_f^2-xyq^2}$$

$$\simeq -2ieQ_f\frac{(Q_fe)^2}{16\pi^2}\,dy\,\frac{(1-y)q^2+m_f^2(1-\cdots)}{(2ym_f^2-yq^2)}\ln\left(\frac{(2ym_f^2-yq^2)x_*+y^2m_f^2}{y^2m_f^2}\right)$$

If $q^2 \gg m_f^2$ ($e^+e^-\to\gamma^*\to f+\bar{f}$; $q^2=s$)

$$\simeq 2ieQ_f\frac{(Q_fe)^2}{16\pi^2}\,dy\,\frac{1-y}{y}\ln\left(\frac{-q^2}{ym_f^2}\right)$$

{ large log if massive
{ log divergence if massless

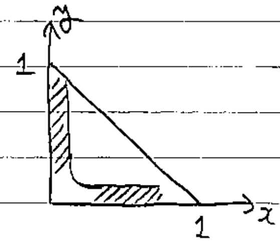
☆ fixed x: integral over small y region

the same.

☆ QCD correction

$\boxed{(Q_fe)^2}$ is replaced by $\boxed{C(R)g^2 = \frac{4}{3}g^2}$ for $q\bar{q}$.

☆ integral over the small-x small-y region



$$(\text{1-loop } \gamma\gamma) \simeq -2ieQ_f \frac{(Q_f e)^2}{16\pi^2} \int dx\,dy \, \frac{q^2 - 2m_f^2}{(x+y)^2 m_f^2 - xy\, q^2}$$

even for $m_f \neq 0$. $\int_0 d\lambda\, \lambda \frac{1}{\lambda^2}$ Log. divergence.

Even after removing the UV divergence by renormalization,

$\boxed{\text{divergence remains in the amplitudes.}}$

Origin of those divergences

$$\Gamma^{\mu}_{(1)} \sim \int \frac{d^4k}{(2\pi)^4} \int dx\,dy\,dz \, \frac{2\delta(x+y+z-1) \quad \overbrace{\left[p^{\nu}, k^{\nu}, \bar{p}^{\nu}, m_f \text{ etc.} \right]}^{(\text{no } x, y \text{ here})}}{\left\{ x\left[(p-k)^2 - m_f^2\right] + y\left[(\bar{p}+k)^2 - m_f^2\right] + z\, k^2 \right\}^3} =: D(k)^3$$

If there is any divergence from integral over a $|k| <$ finite region,

there must be (x_*, y_*, z_*), k^{μ}_* where

- $D((x,y,z)_*, k^{\mu}_*) = 0$
- $\left[\frac{\partial}{\partial k^{\mu}} D\right]((x,y,z)_*, k^{\mu}_*) = 0$

} double root of $D(k^{\mu}) = 0$. otherwise the integration contour can be deformed to avoid $D(k) = 0$.

- $(p-k_*)^2 - m_f^2 = 0$ or $x_* = 0$ or $x_* = 1$
- $(\bar{p}+k_*)^2 - m_f^2 = 0$ or $y_* = 0$ or $y_* = 1$
- ~~$k^2 = 0$~~ or ~~$z_* = 0$ or $z_* = 1$~~

} so that $D(k) = 0$ cannot be ~~avoided by deforming the~~ integration contour of $x, y, (z)$

OK to drop this option $\left(x_* = 1 \Rightarrow y_* = z_* = 0 \Downarrow D(k_*) = 0,\ (p-k_*)^2 - m_f^2 = 0 \text{ automatic} \right)$

$(x_*, y_*, z_*, k^{\nu}_*)$ satisfying all the conditions above.

$\Longrightarrow$ $\boxed{\text{pinch (hyper) surface}}$

☆ $\boxed{x_* = y_* = 0 \ \& \ k_*^2 = 0}$ $\quad x_*(k_* - p)^\mu + y_*(k_* + \bar{p})^\mu + z_* k_*^\mu = 0 \Rightarrow \boxed{k_*^\mu = 0}$

Introduce a scaling parameter $\lambda \ll 1$ in the (x, y, k^ν) space around $(x_*, y_*, k_*^\nu) = (0, 0, 0^\nu)$.

$$(x, y, k^\nu) \sim (\lambda x_0, \lambda y_0, \lambda k_0^\nu)$$

- $dx\, dy\, d^4k \sim d^5\Omega\, d\lambda\, \lambda^5$
- $[D(x, y, k)]^3 \cong \Big(\lambda x_0 \cdot \underbrace{[(p-k)^2 - m_f^2]}_{(p^2 - m_f^2) - 2p\cdot k + k^2} + \lambda y_0 \cdot \underbrace{[(\bar{p}+k)^2 - m_f^2]}_{(\bar{p}^2 - m_f^2) + 2\bar{p}\cdot k + k^2} + z \cdot \underbrace{k^2}_{\lambda^2 k_0^2}\Big)^3 \sim \lambda^6$

  $(p^2 - m_f^2) = 0$, $2p\cdot k \sim \lambda$, $k^2 \sim \lambda^2$; $(\bar{p}^2 - m_f^2) = 0$, $2\bar{p}\cdot k \sim \lambda$, $k^2 \sim \lambda^2$

$$\text{so} \int d^5\Omega \left(\int_0 \frac{d\lambda\, \lambda^5}{\lambda^6} \sim \text{log divergence} \right)$$

associated with $k^\nu \sim (k_*^\nu = 0^\nu)$ $\underline{\underline{\text{soft}}}$ $\left\{ \begin{matrix} \text{photon} \\ \text{gluon} \end{matrix} \right\}$.

☆ $\boxed{x_* = 0 \ \& \ (\bar{p}+k)^2 - m_f^2 = 0 \ \& \ k^2 = 0}$ $\quad \overset{\nearrow 0}{x_*}(k_* - p)^\mu + y_*(k_* + \bar{p})^\mu + z_* k^\mu = 0$

$$\Rightarrow \frac{(-k_*^\mu)}{y_*} = \frac{\bar{p}^\mu - (-k_*)^\mu}{(1 - y_*)}$$

$k_*^\mu \parallel [\bar{p} - (-k_*)]^\mu$ and both on-shell

$\longrightarrow$ possible only if $m_f = 0$.

$\left(\begin{matrix} \text{photon/gluon} \\ \underline{\underline{\text{collinear to a massless } \bar{f}.}} \end{matrix} \right)$

$\left\{ \begin{matrix} (-k_*^\mu) = y_*(\bar{p}^\mu) \\ [\bar{p} - (-k_*)]^\mu = (1 - y_*)(\bar{p}^\mu) \end{matrix} \right\}$

light cone components of a four vector

$$\left(\frac{l^0 + l^3}{\sqrt{2}}, \frac{l^0 - l^3}{\sqrt{2}}, \vec{l}_T \right) =: (l^+, l^-, \vec{l}_T)$$

Introduce a scaling parameter $\lambda \ll 1$.

$$\bar{p}^\mu = (E, -E, \vec{0}) \qquad (-k^\mu) = \left(y_0 E + \frac{k_0^- + \lambda^2 k_0^+}{\sqrt{2}}, -y_0 E - \frac{-k_0^- + \lambda^2 k_0^+}{\sqrt{2}}, -\lambda \vec{k}_{T,0} \right) \quad x = \lambda^2 x_0$$

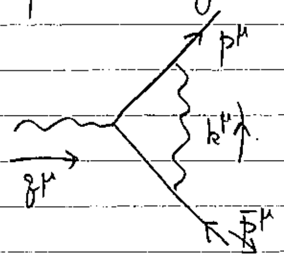
$$\Rightarrow (\bar{p}+k)^2 - m_f^2 \simeq O(\lambda^2) \qquad k^2 \simeq O(\lambda^2) \qquad \text{so } \underline{D(x, y, k) \sim O(\lambda^2)}$$

- $dx\, dy\, d^4k \sim [dk_0^-\, dy_0] \left[dk^+ d^2\vec{k}_T\, dx \sim d^3\Omega\, d\lambda\, \lambda^5 \right]$

$$\text{so} \quad dy_0\, dk_0^- \int d^3\Omega \left(\int \frac{d\lambda\, \lambda^5}{\lambda^6} \sim \text{log divergence} \right)$$

$\boxed{\text{Recap}}$

<u>soft divergence</u>

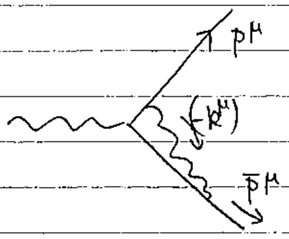


$k^\mu \approx 0$

(soft)

$$\left.\begin{array}{l} \dfrac{(\cancel{p}-\cancel{k})+m}{(p-k)^2-m_f^2} \\[2ex] \dfrac{-(\cancel{\bar{p}}+\cancel{k})+m}{(\bar{p}+k)^2-m_f^2} \end{array}\right\} \Rightarrow \boxed{\begin{array}{l}\text{nearly on-shell} \\ \text{(small virtuality)}\end{array}}$$

<u>collinear divergence</u> (for massless fermion)

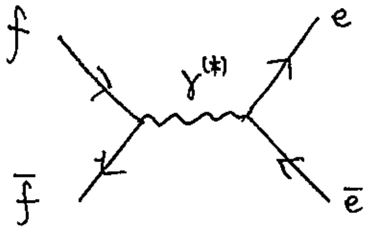


$$\left\{\begin{array}{l} (-k^\mu) \\ [\bar{p}-(-k^\mu)]^\mu \end{array}\right. \begin{array}{l}\text{almost parallel} \\ \quad\text{almost on-shell}\end{array} \Leftarrow \boxed{\begin{array}{l}\text{nearly on-shell} \\ \text{intermediate} \\ \text{states}\end{array}}$$

(collinear)

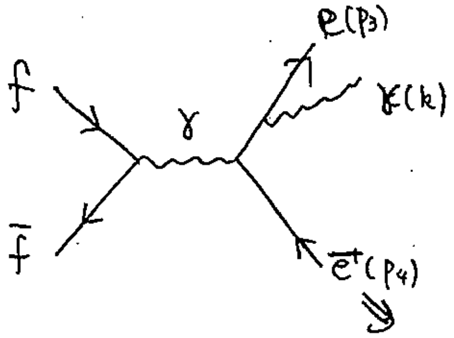
§9.2 Divergence in Real Emission

$f+\bar{f} \to e^+ + e^- + \gamma$ 3-body final state. in QED s-channel.



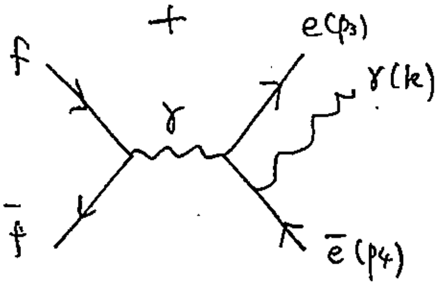
$$\Rightarrow \quad i\mathcal{M} = \left[ie\,\bar{v}(p_2)\gamma_\mu u(p_1)\right]\frac{-i}{s}\left[\bar{u}(p_3)\gamma^\mu v(p_4)\ ie\right]$$

$$\sigma(f+\bar{f}\to e^- + e^+) \propto \int\frac{d^3\vec{p}_3}{(2\pi)^3}\frac{1}{2E_{p_3}}\int\frac{d^3\vec{p}_4}{(2\pi)^3}\frac{1}{2E_{p_4}}(2\pi)^4\delta^4(p^\mu_{in}-p^\mu_{out}) \times |\mathcal{M}|^2$$



$$\left[ie\,\bar{u}(p_3)(i\gamma^\lambda e)\frac{i[(\not{p}_3+\not{k})+m_e]}{(p_3+k)^2-m_e^2+i\varepsilon}\gamma^\mu v(p_4)\right]$$

+



ignore $\not{k}$, m_e and keep $\not{p}_3$ in [ ]

$$\begin{cases} \gamma^\lambda \not{p}_3 = \{\gamma^\lambda, \not{p}_3\} - \not{p}_3\gamma^\lambda = 2p_3^\lambda - \not{p}_3\gamma^\lambda \\ \bar{u}(p_3)\cdot[\not{p}_3 - m_e] = 0 \quad \text{(Dirac eq.)} \\ (\Rightarrow \bar{u}(p_3)\not{p}_3 \simeq 0) \end{cases}$$

(eikonal approximation) $\Leftarrow$

$$\left[(ie)\cdot\bar{u}(p_3)\gamma^\mu v(p_4)\right]\times\frac{2ie\,p_3^\lambda}{2p_3\cdot k+i\varepsilon}$$

similarly

$$\left[(ie)\ \bar{u}(p_3)\gamma^\mu v(p_4)\right]\times\frac{-2ie\,p_4^\lambda}{2p_4\cdot k+i\varepsilon}$$

$$\Rightarrow \boxed{\left|\mathcal{M}^\kappa_{e\bar{e}\gamma}\,\varepsilon^{(*)}_\kappa\right|^2 \simeq \varepsilon_\kappa(k)\,\varepsilon^*_\lambda(k)\left(\frac{p_3^\kappa}{p_3\cdot k}-\frac{p_4^\kappa}{p_4\cdot k}\right)\left(\quad\right)^\lambda(e^2) \times |\mathcal{M}_{e\bar{e}}|^2}$$

$k^2=0$ (γ : on-shell).

γ spin sum (polarization) $\Rightarrow$ $[\varepsilon_\kappa(k)\,\varepsilon^*_\lambda(k) \Rightarrow -\eta_{\kappa\lambda}.]$.

$$\Rightarrow |\mathcal{M}_{e\bar{e}}|^2\times e^2\left\{\frac{2(p_3\cdot p_4)}{(p_3\cdot k)(p_4\cdot k)} - \underbrace{\frac{m_e^2}{(p_3\cdot k)^2} - \frac{m_e^2}{(p_4\cdot k)^2}}\right\}$$

$\hookrightarrow$ power suppressed. $\to$ ignore.

$$\sigma(f+\bar f \to e+\bar e+\gamma) \;\; \cancel{\sigma(f+\bar f \to e+\bar e)} \times$$

$$\propto \int \frac{d^3\vec p_3}{(2\pi)^3}\frac{1}{2E_3} \int \frac{d^3\vec p_4}{(2\pi)^3}\frac{1}{2E_4} \int \frac{d^3\vec k}{(2\pi)^3}\frac{1}{2E_k} (2\pi)^4 \delta^4(p_{in}^\mu - p_{out}^\mu) |\mathcal{M}_{ee\gamma}|^2$$

$$= \sigma(\to e+\bar e) \int \frac{d^3\vec k}{(2\pi)^3}\frac{1}{2E_k} \times (2e^2)\frac{(p_3\cdot p_4)}{(p_3\cdot k)(p_4\cdot k)}$$

center of mass frame of $f+\bar f$ collision.

$$\Rightarrow \quad p_3^\mu \sim E(1,\vec v)$$
$$p_4^\mu \sim E(1,\vec v') \qquad \vec v' \simeq -\vec v \qquad \frac{(p_3\cdot p_4)}{(p_3\cdot k)(p_4\cdot k)} = \frac{(1-\vec v\cdot\vec v')}{k^2(1-\vec n\cdot\vec v)(1-\vec n\cdot\vec v')}$$
$$k^\mu \sim k(1,\vec n).$$

$$\boxed{\sigma(\to e\bar e\gamma) \simeq \sigma(\to e\bar e)\frac{e^2}{(2\pi)^3}\int\frac{dk}{k}\int d\Omega_{\vec n}\,\frac{(1-\vec v\cdot\vec v')}{(1-\vec n\cdot\vec v)(1-\vec n\cdot\vec v')}}$$

from a region $\vec n$ almost parallel to $\vec v$ (angle θ.

$$\frac{e^2}{(2\pi)^3}\int\frac{dk}{k}(2\pi)\int^1 d\cos\theta\,\frac{1}{1-\cos\theta|v|} \qquad |v| = \frac{|p_3|}{E} \simeq 1-\frac{m_e^2}{2E^2}$$

$$\hookrightarrow 1-\left(1-\frac{m_e^2}{2E^2}\right)\cos\theta.$$

$$\simeq \frac{e^2}{(2\pi)^2}\int\frac{dk}{k}\left(-\ln\left(\frac{m_e^2}{2E^2}\right)\right) = \frac{e^2}{(2\pi)^2}\int\frac{dk}{k}\ln\left(\frac{s}{2m_e^2}\right)$$

$$* \boxed{\sigma(\to e\bar e\gamma) \simeq \sigma(\to e\bar e)\times\left[\frac{e^2}{4\pi^2}\int\frac{dk}{k}\ln\left(\frac{s}{2m_e^2}\right)\right]}$$
$|k_\gamma| \ll \sqrt{s}$.
$\vec n_\gamma$ almost $/\!/ \ \vec p_3$

collinear divergence $\left\{\begin{array}{l}\gamma \text{ emission } /\!/ \text{ to } e^- \\ \quad " \qquad /\!/ \text{ to } e^+\end{array}\right\}$ if $(m_e \ll s) \Leftarrow \left\{\begin{array}{l}\frac{1}{p_3\cdot k}\sim\frac{1}{0} \\ \frac{1}{p_4\cdot k}\sim\frac{1}{0}\end{array}\right.$ $(k^2=0)$

soft divergence propagator is nearly on-shell.

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§ 10 Cancellation of IR divergence.

Observation

- soft divergence : massless ~~state~~ particle (γ) arbitrarily low energy

$\Rightarrow$ can we see it?

- collinear divergence : kinematically possible for massless particles.

$$p^\mu \to \lambda_1 p^\mu + (1-\lambda_1)p^\mu \to \lambda_1 p^\mu + \lambda_2 p^\mu + (1-\lambda_1-\lambda_2)p^\mu$$
$$\to \cdots$$
$$\to \sum_i (\lambda_i p^\mu) \quad \text{so that } \left(\sum_i \lambda_i = 1\right).$$

can we distinguish them?

$|\vec{k}_\gamma| \ll \sqrt{s}$

$\vec{k}_\gamma // \vec{p}_e$ or $\vec{p}_{\bar{e}}$ part of $\sigma(\to e\bar{e}\gamma)$ should be treated as a part of $\sigma(\to e\bar{e})$.

Look at the collinear part

$$\left\{\begin{array}{l} \bullet\ \sigma(\to e\bar{e}) \simeq \sigma(\to e\bar{e})_{\text{tree}} \times \left|1 - \frac{e^2}{8\pi^2}\int dy \frac{1-y}{y} \ln\left(\frac{-s}{m_e^2}\right)\right|^2 \\ \qquad \simeq \sigma(\to e\bar{e})_{\text{tree}} \times \left[1 - \frac{e^2}{4\pi^2}\int dy \frac{1}{y} \ln\left(\frac{s}{m_e^2}\right)\right] \quad \left(\begin{array}{c}\text{approx.}\\ 1-y \simeq 1.\end{array}\right) \\ \bullet\ \sigma(\to e\bar{e}\gamma) \simeq \sigma(\to e\bar{e})_{\text{tree}} \times \left[+ \frac{e^2}{4\pi^2}\int \frac{dk}{k} \ln\left(\frac{s}{2m_e^2}\right)\right]. \end{array}\right.$$

$$\ln\left(\frac{1}{2m_e^2/s}\right) \Rightarrow \ln\left(\frac{1-\cos\theta_*}{2m_e^2/s}\right) + \ln\left(\frac{1}{1-\cos\theta_*}\right)$$

$$\boxed{\sigma(\to e\bar{e}\gamma)_{|k|>k_*,\ \theta>\theta_*} \simeq \sigma(\to e\bar{e}) \times \left[\frac{e^2}{4\pi^2}\int_{k_*} \frac{dk}{k} \ln\left(\frac{1}{1-\cos\theta_*}\right)\right]}$$

$\Leftarrow$ positive

$$\boxed{\begin{array}{l}\sigma(\to e\bar{e}) \\ +\ \sigma(\to e\bar{e}\gamma)_{|k|>k_*,\ \theta_*>\theta}\end{array} \simeq \sigma(\to e\bar{e}) \times \left[1 - \frac{e^2}{4\pi^2}\int_{2k_*/\sqrt{s}} \frac{dy}{y} \ln\left(\frac{2}{1-\cos\theta_*}\right)\right]}$$

negative correcti[on]

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