§10.2 Cutkowsky rule

☆ Unitarity of the S-matrix

$$S_{\beta\alpha} =: (\mathbb{1}_{\beta\alpha} + i\mathcal{M}_{\beta\alpha}), \qquad \boxed{S^\dagger S = \mathbb{1}}$$

$$\Longleftrightarrow (\mathcal{M}^\dagger)_{\gamma\beta}(\mathcal{M})_{\beta\alpha} + i\mathcal{M}_{\gamma\alpha} - i(\mathcal{M}^\dagger)_{\gamma\alpha} = 0$$

$$\Longleftrightarrow \boxed{\left[\mathcal{M}_{\gamma\alpha} - (\mathcal{M}_{\alpha\gamma})^*\right]/i = \sum_\beta (\mathcal{M}_{\beta\gamma})^*(\mathcal{M}_{\beta\alpha})}$$

$$\left(\text{for } |\gamma\rangle = |\alpha\rangle : \quad 2\,\mathrm{Im}(\mathcal{M}_{\alpha\alpha}) = \sum_\beta |\mathcal{M}_{\beta\alpha}|^2 \quad \text{optical thm}\right)$$

☆ Analyticity of scattering amplitudes (graph by graph)

$$\left[i\mathcal{M}^{\Gamma}_{\gamma\alpha}(s+i\epsilon) - i\mathcal{M}^{\Gamma}_{\gamma\alpha}(s-i\epsilon)\right] \overset{ⓐ}{=} \sum_{\substack{\text{any}\\ \text{cut in the}\\ s\text{-channel}}} \left(i\mathcal{M}^{\Gamma_R}_{\gamma\beta}\right)\left(i\mathcal{M}^{\Gamma_L}_{\beta\alpha}\right)$$

$$\overset{ⓑ}{=} \sum_{\text{any cut}} -\left(i\mathcal{M}^{\overleftrightarrow{\Gamma_R}}_{\beta\gamma}\right)^*\left(i\mathcal{M}^{\Gamma_L}_{\beta\alpha}\right)$$

[discontinuity in the s-channel] is from propagators.

ⓐ: See Sterman's lecture note appendix B (time-ordered perturbation theory is used)

Contributions on the RHS of ⓐ come only from cuts where all the cut propagators can be on-shell simultaneously.

$\Longrightarrow$ in a cut graph, replace all the cut propagators by

$$\frac{i}{p^2 - m^2 + i\epsilon} \longrightarrow \left\{\frac{i}{(p^2-m^2+i\epsilon)} - \frac{i}{(p^2-m^2-i\epsilon)}\right\} \simeq (2\pi)\delta^+(p^2-m^2) = \frac{(2\pi)}{(2E_{\vec{p}})}\delta(E - E_{\vec{p}})$$

for use in the RHS of ⓐ

ⓑ: • $\mathcal{L}_{int} \supset c_j \mathcal{O}_j$ is use for $(i\mathcal{M}^{\Gamma}_{\gamma\beta}) \Longleftrightarrow \mathcal{L}_{int} \supset c_j^* \mathcal{O}_j^\dagger$ is used for $(i\mathcal{M}^{\overleftrightarrow{\Gamma}}_{\beta\gamma})$

• $\underset{\text{in } (i\mathcal{M})}{\left[\#\text{ of extra}(\pm i)\text{'s}\right]} = (\# I) + (\# V) + (\# L) \underset{\substack{\boxed{\text{mod } 2}\\ (\mathbb{R} \text{ or not})}}{\equiv} \#(\text{connected components}) = 1$

So, $$\boxed{\left[\mathcal{M}^{\Gamma}_{\gamma\alpha}(s+i\varepsilon) - \mathcal{M}^{\Gamma}_{\gamma\alpha}(s-i\varepsilon)\right]\Big/ i = \sum_{\substack{\text{any cut}\\ \text{in } s\text{-ch}}} \left(\mathcal{M}^{(\overleftrightarrow{\Gamma_R})}_{\beta\gamma}\right)^* \left(\mathcal{M}^{\Gamma_L}_{\beta\alpha}\right)}$$

computed by the Cutkowsky rule

(sum over all possible cuts
and the replacement of all the cut propagators)

sum over graphs

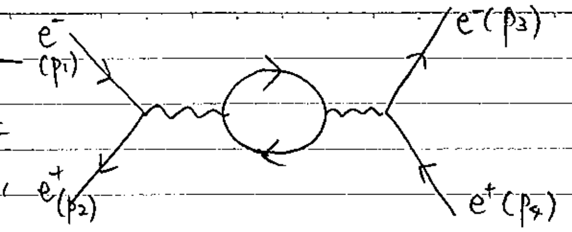
$$\boxed{\left[\mathcal{M}_{\gamma\alpha}(s+i\varepsilon) - \mathcal{M}_{\gamma\alpha}(s-i\varepsilon)\right]\Big/ i = \sum_{\beta} \left(\mathcal{M}_{\beta\gamma}\right)^* \mathcal{M}_{\beta\alpha}}$$

(Combine both the unitarity and the analyticity to obtain.

$$\boxed{\left[\mathcal{M}_{\gamma\alpha}(s+i\varepsilon) - \mathcal{M}_{\gamma\alpha}(s-i\varepsilon)\right]\Big/ i = \left[\mathcal{M}_{\gamma\alpha} - \left(\mathcal{M}_{\alpha\gamma}\right)^*\right]\Big/ i}$$

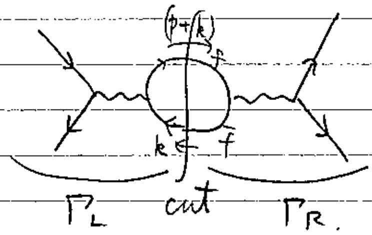
they are not necessarily $\mathbb{R}$-valued if $|\gamma\rangle \neq |\alpha\rangle$.)

An example (graph Γ) =



$$i\mathcal{M}^{\Gamma}_{\beta\alpha} = \left(+ie\left[\bar{v}(\vec{p}_2)\gamma^\mu u(\vec{p}_1)\right]\right)\left(ie\left[\bar{u}(\vec{p}_3)\gamma^\nu v(\vec{p}_4)\right]\right)\left(\frac{-i}{s+i\varepsilon}\right)^2 is\,\eta_{\mu\nu}\,\Pi^{(1)}_{ren}(s)$$

- single photon in the s-channel cannot be on-shell.
- The only possible cut of this graph is



$f: \quad (p_{CM}+k)^\mu = ((\sqrt{s}+k^0), \vec{k})$

$\bar{f}: \quad (-k)^\mu = ((-k^0), -\vec{k})$

both are on-shell $\Longleftrightarrow$ $(-k^0) = \sqrt{|\vec{k}|^2+m_f^2}$ and

$\sqrt{s}+k^0 = \sqrt{|\vec{k}|^2+m_f^2}$

The discontinuity $\left[i\mathcal{M}^{\Gamma}(s+i\varepsilon) - i\mathcal{M}^{\Gamma}(s-i\varepsilon)\right]$ can arise only from the discontinuity $\left[i\Pi^{(1)}_{ren}(s+i\varepsilon) - i\Pi^{(1)}_{ren}(s-i\varepsilon)\right]$.

◎ Cutkowsky rule

$$\left(i\Pi^{(1)}_{ren}(s+i\varepsilon) - i\Pi^{(1)}_{ren}(s-i\varepsilon)\right)\times s\,\eta_{\mu\nu} \sim \int\frac{d^4k}{(2\pi)^4}\frac{1}{2E_f}\frac{1}{2E_{\bar{f}}}(2\pi)^2\delta(\sqrt{s}+k^0-E_f)\,\delta(-k^0-E_{\bar{f}})$$

$$(-)\,\mathrm{Tr}\left[\gamma^\mu(\not{k}+m_f)\gamma^\nu((\not{p}_{CM}+\not{k})+m_f)\right](-ieQ_f)^2$$

$$= \frac{-e^2Q_f^2}{s\,(2\pi)^2}\int d|\vec{k}|\,|\vec{k}|^2\int d^2\Omega\;\delta(\sqrt{s}-2E_{\vec{k}})\;4\left[k_{\bar{f}}^\mu k_f^\nu + k_f^\mu k_{\bar{f}}^\nu - \eta^{\mu\nu}\{(k_{\bar{f}}\cdot k_f)+m_f^2\}\right]$$

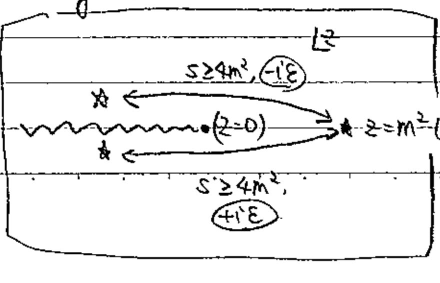
$$\sim \frac{-e^2Q_f^2}{s\,(2\pi)^2}\frac{|k|E_{\vec{k}}}{2}\,4\times\left[-(4\pi)\frac{s+2m_f^2}{3}\eta^{\mu\nu}\right]$$

$$= \boxed{+\frac{e^2Q_f^2}{\pi}\frac{s+2m_f^2}{6s}\sqrt{1-\frac{4m_f^2}{s}}}\times s\,\eta_{\mu\nu}$$

◎ We have computed directly.

$$\boxed{\Pi^{(1)}_{ren}(s+i\varepsilon) = \frac{e^2Q_f^2}{2\pi^2}\int_0^1 dx\; x(1-x)\ln\left(\frac{m_f^2-x(1-x)(s+i\varepsilon)}{m_f^2+x(1-x)\mu^2}\right)}$$

$\log(z)$ with $z = m^2 - x(1-x)(s\pm i\varepsilon)$



$$\left[i\Pi_{ren}(s+i\varepsilon) - i\Pi_{ren}(s-i\varepsilon)\right] = \frac{e^2Q_f^2}{2\pi^2}\,i\cdot(-2\pi i)\int_{x_-}^{x_+}dx\;x(1-x)$$

$$= \frac{e^2Q_f^2}{\pi}\left\{\frac{(x_+^2-x_-^2)}{2} - \frac{(x_+^3-x_-^3)}{3}\right\} = \boxed{\frac{e^2Q_f^2}{\pi}\frac{s+2m_f^2}{6s}\sqrt{1-\frac{4m_f^2}{s}}}$$

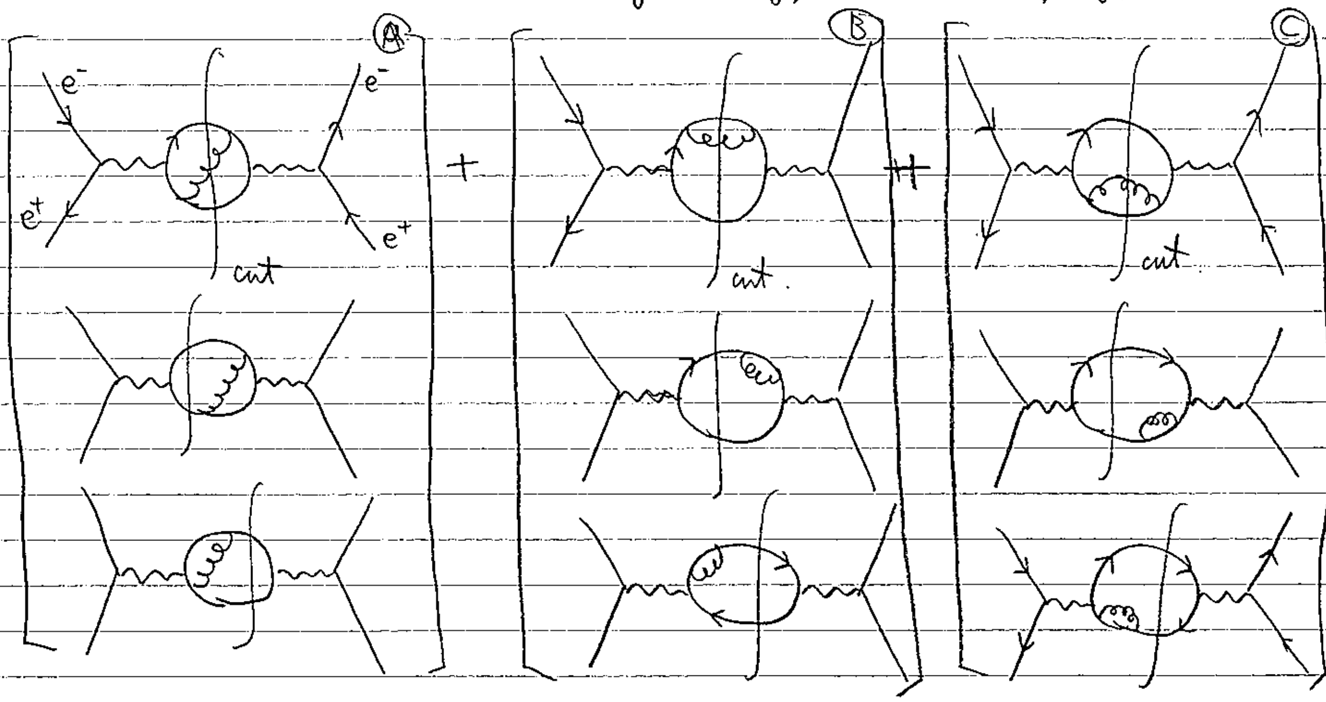
$$x_\pm = \frac{1}{2}\left(1\pm\sqrt{1-\frac{4m_f^2}{s}}\right)$$

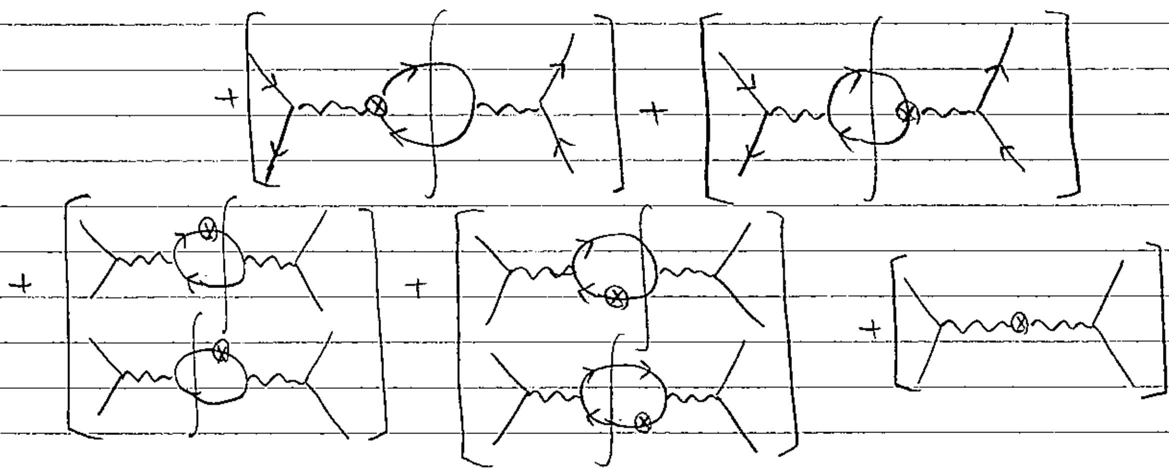
The optical theorem (+ analyticity)

$$\boxed{\sigma(e^- e^+ \to \text{anything}) = \frac{1}{2s}\left(\left[\mathcal{M}_{\alpha\alpha} - \mathcal{M}^*_{\alpha\alpha}\right]/i\right) = \frac{-1}{2s}\left(i\mathcal{M}_{\alpha\alpha}(s+i\varepsilon) - i\mathcal{M}_{\alpha\alpha}(s-i\varepsilon)\right)}$$

$\mathcal{M}(e^- e^+ \to ⊘ \to e^- e^+)$ at $\mathcal{O}(e^4 \sim \alpha_e^2) \Rightarrow \sigma(e^- e^+ \to f\bar{f})$ tree level

$\mathcal{M}(e^- e^+ \to ⊘ \to e^- e^+)$ at $\mathcal{O}(e^4 g^2 \sim \alpha_e^2 \alpha_g) \Rightarrow \sigma(e^- e^+ \to q\bar{q}g)$ NLO





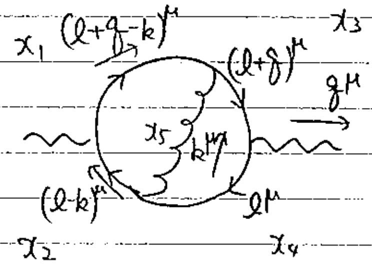
The discontinuity in

$(i\mathcal{M}_{\alpha\alpha})(s \pm i\varepsilon)$ @ $\mathcal{O}(e^4 g^2) \sim \alpha_e^2 \alpha_g$ $\Longleftrightarrow$ $\left[\begin{array}{l} \sigma(e^- e^+ \to q\bar{q}) \text{ incl. } \alpha_s \text{ correction} \\ \quad + \\ \sigma(e^- e^+ \to q\bar{q}g) \end{array}\right]$

$\left[\exists ?\ \text{well-defined limit even if } m_q = m_g = 0\right]$ $\Longleftrightarrow$ $\left[\exists ?\ \text{well-defined limit even if } m_q = 0,\ m_g = 0\right]$

Does $iM =$  have a well-defined $m_q = m_g = 0$ limit?

Ⓐ



Step 1

- for each i : either $x_i = 0$ or $(l_i^2 - m_i^2) = 0$
- for each loop momentum l_a^μ

$$\frac{\partial}{\partial l_a^\mu}\left(\sum_i x_i (l_i^2 - m_i^2)\right)^{=D} = 0 \quad \text{——} (*)$$

$\Leftrightarrow$ pinch singularity surface

Step 2 power counting around a pinch surface.

$$D = x_1 (l+q-k)^2 + x_2 (l-k)^2 + x_3 (l+q)^2 + x_4 l^2 + x_5 k^2$$

eg. $x_1 \sim 0 \quad x_2 \sim 0 \quad x_3 \sim 0 \quad x_4 \sim 0 \quad (x_5 \sim 1)\ \&\ k^2 \sim 0$

$(*\text{for } l^\mu) \Rightarrow$ trivial $\qquad (*\text{for } k^\mu) \Rightarrow x_5 k^\mu = 0$

power counting $\quad k^\mu \sim O(\lambda), \quad x_{1,2,3,4} \sim O(\lambda^2)$

measure $\Rightarrow d^4k\, d^4x \sim O(\lambda^4 \cdot \lambda^8 = \lambda^{12})$
denominator $\Rightarrow D^5 \sim O(\lambda^{2\cdot 5} = \lambda^{10})$
$\Big\}\ \int_0 \frac{d\lambda^{12}}{\lambda^{10}}$ is finite.

eg $x_1 \sim 0 \quad x_2 \sim 0, \quad (l+q)^2 \sim 0, \quad l^2 \sim 0, \quad k^2 \sim 0.$

$(*\text{ for } k^\mu) \Rightarrow x_5 k^\mu = 0 \qquad (*\text{ for } l^\mu) \Rightarrow x_3 (l+q)^\mu + x_4 l^\mu = 0$

Impossible : $(l+q)^\mu$ and l^μ are parallel, light-like and $(l+q)^\mu - l^\mu = q^\mu$ is time-like.

So, such a component of the pinch surface is absent.

Complete analysis : $M_{\alpha\alpha}^{Ⓐ}$ has a well-defined limit, and so do $M^{Ⓑ}$, $M^{Ⓒ}$.

2-loop caluction $\quad \sigma_{tot} \simeq \frac{4\pi\alpha_e^2}{3S}(N_c Q_q^2)\left[1 + C_2 \frac{3\alpha_s(\mu)}{4\pi} + \cdots\right]$

$\Updownarrow$

$$\sigma(\to q\bar{q}) \cong \frac{4\pi\alpha_e^2}{3S}(N_c Q_q^2)\left(1 - C_2\frac{\alpha_s}{\pi}\left\{\int_\varepsilon^1 \frac{dy}{y}\ln\left(\frac{S}{m_q^2}\right) + \int_\varepsilon^1 \frac{dx}{x}\ln\left(\frac{S}{m_q^2}\right) + \int_0^\varepsilon \frac{dx}{x}\int_0^\varepsilon \frac{dy}{y}\right\} + (\text{finite})\right)$$

$$\sigma(\to q\bar{q}g) \simeq \frac{4\pi\alpha_e^2}{3S}(N_c Q_q^2)\left(C_2\frac{\alpha_s}{\pi}\left\{\int_{m_q}^{\sqrt{s}} \frac{dk}{k}\ln\left(\frac{S}{m_q^2}\right) + \int_{m_q}^{\sqrt{s}} \frac{dk}{k}\ln\left(\frac{S}{m_q^2}\right) + \int_0^{m_g/\sqrt{s}} \frac{dx}{x}\int_0 \frac{dy}{y}\right\} + (\text{finite})\right)$$

§10.3 OPE and Non-perturbative corrections

$e^- + e^+ \to q\bar{q}g$ etc. : hadron production in fact.

In which sense is perturbative QCD approach justified for $s \gg \Lambda_{QCD}^2$?

The optical thm : insertion of a complete system of the Hilbert space either in q, g or in hadron

does not rely on perturbation.

But $\sigma_{tot}(s) \propto e^2 \left[i\Pi(s+i\varepsilon) - i\Pi(s-i\varepsilon) \right]$ and we need to evaluate

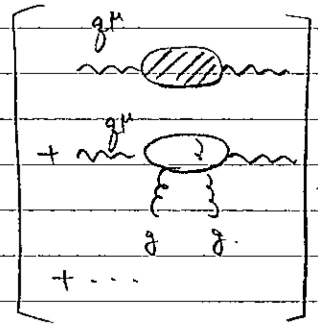
$$\boxed{(2\pi)^4 \delta^4(q-q')\, i(q^2\eta^{\mu\nu} - q^\mu q^\nu)\, \Pi(q^2+i\varepsilon) = (-ie)^2 \iint \langle\Omega| T\{J^\mu(x) J^\nu(y)\} |\Omega\rangle\, e^{iq'\cdot x}\, e^{-iq\cdot y}\, d^4x\, d^4y}$$

Remember that the operator product expansion (OPE) expands

$$(-ie)^2 \int d^4x\, T\{J^\mu(x) J^\nu(0)\}\, e^{iq\cdot x} = i(q^2\eta^{\mu\nu} - q^\mu q^\nu)\, \Pi(q^2+i\varepsilon)\, \mathbb{1}$$

$$+\ i(\quad \text{〃} \quad) \times \left[\mathcal{O}\left(\frac{1}{(q^2)^2}\right) \right] \mathrm{tr}(F_{\rho\sigma}F^{\rho\sigma})(0)$$

$$+\ \cdots\cdots$$



$\mathrm{tr}[F_{\rho\sigma}F^{\rho\sigma}]$

☆ Is OPE defined at non-perturbative level?

maybe if local operators are defined at non-perturbative level.

☆ The perturbative computation $\Pi_{pert}(q^2)$ for $\Pi(q^2)$

may have zero convergence radius in α_s

still $\left\{ \left(\text{Borel resummation of } \Pi_{pert}\right) + \text{minimum non-P contributions to add} \right\}$ may become well-defined. $=: [\Pi_{pert}]_B$

☆ also quite likely that $\langle\Omega| \mathrm{tr}(F_{\rho\sigma}F^{\rho\sigma}) |\Omega\rangle \sim (\Lambda_{QCD})^4 \neq 0$.

This will give rise to (yet another) contribution supp. relatively by $\left(\frac{\Lambda_{QCD}^4}{s^2}\right)$.

from Peskin–Schroeder $\boxed{\dfrac{\sigma(e^-e^+\to \text{hadron})}{\left[\sigma(e^-e^+\to\mu^-\mu^+)=\frac{4\pi\alpha_e^2}{3s}\right]} =: R(s)}$ data vs theory

$\cdot\, R = R_0\cdot K.$ $\boxed{R_0 = \sum_f N_c Q_f^2}$ $\boxed{K \simeq 1 + \frac{\alpha_s(\mu)}{\pi} + \cdots}$

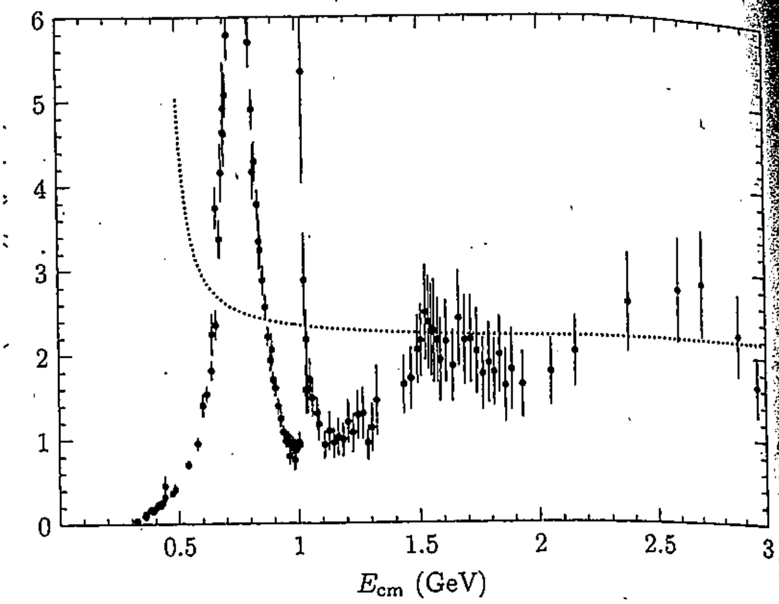
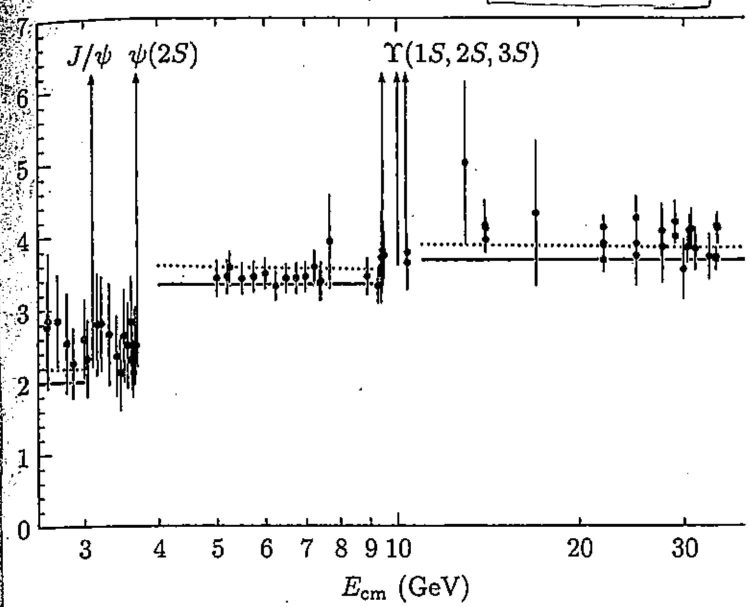


Figure 18.8. Experimental measurements of the total cross section for the reaction $e^+e^-\to$ hadrons at energies below 3 GeV, compared to the prediction of perturbative QCD for 3 quark flavors. The data are taken from the compilation of M. Swartz, *Phys. Rev.* D53, 5268 (1996). Complete references to the various results are given there.

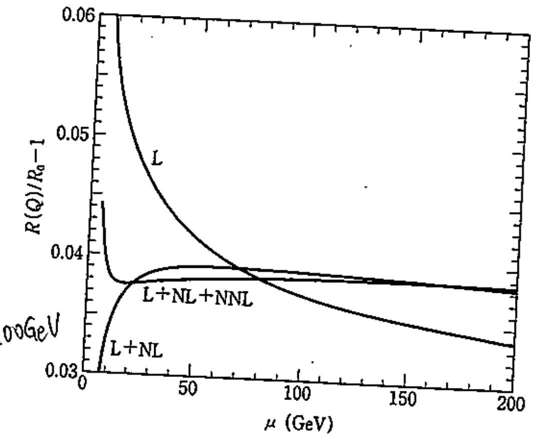


igure 5.3. Experimental measurements of the total cross section for action $e^+e^-\to$ hadrons, from the data compilation of M. Swartz, P ev. **D53**, 5268 (1996). Complete references to the various experiments ven there. The measurements are compared to theoretical predictions f uantum Chromodynamics, as explained in the text. The solid line is mple prediction (5.16).

:tions, which reflect the form of the proton wavefunction and are det ed by soft QCD dynamics. However, we saw in Section 17.5 that effects D perturbation theory cause the parton distributions to change their fo . function of the m...

$e^+e^-\to\mu^+\mu^-$: Helicity Structure

r a reaction is generally easy to c



$\sqrt{s}=100$GeV

図5.6 $R=\dfrac{\sigma(e^+e^-\to\text{ハドロン})}{\sigma(e^+e^-\to\mu\bar\mu:\text{QED})}$ のQCD効果のスケール依存性[5]
図には $\dfrac{R(Q=100\text{GeV})}{R_0}-1$, $R_0=\dfrac{11}{3}$ を書いてある.
L=対数第1近似 (leading log approximation), NL=第2近似 (next-to-LLA),
NNL=第3近似 (next next-to-LLA).

from Nagashima p.139

By using $\alpha_s(\mu)$ with $\mu\sim\sqrt{s}$
K^L can be a good approximation.