§10.3 OPE and Non-perturbative corrections

$e^- + e^+ \to q\bar{q}g$ etc : hadron production in fact.

In which sense is perturbative QCD approach justified for $s \gg \Lambda_{QCD}^2$?

The optical thm : insertion of a complete system of the Hilbert space either in $q.g.$ or in hadron.

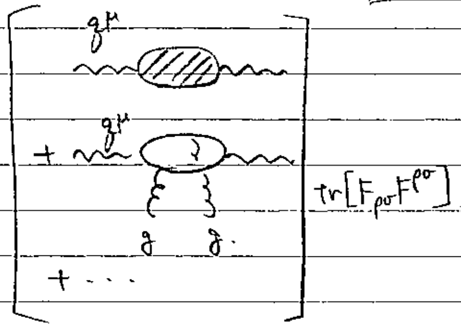
does not rely on perturbation.

But $\sigma_{tot}(s) \propto e^2 \left[i\Pi(s+i\varepsilon) - i\Pi(s-i\varepsilon) \right]$ and we need to evaluate

$$\boxed{(2\pi)^4 \delta^4(q-q')\, i(q^2\eta^{\mu\nu} - q^\mu q^\nu)\, \Pi(q^2+i\varepsilon) = \left(-i\underset{Q_f}{e}\right)^2 \iint \langle\Omega| T\{J^\mu_f(x) J^\nu_f(y)\}|\Omega\rangle\, e^{iq'\cdot x}\, e^{-iq\cdot y}\, d^4x\, d^4y}$$

Remember that the operator product expansion (OPE) expands

$$\boxed{\begin{aligned}\left(-i\underset{Q_f}{e}\right)^2 \int d^4x\, T\{J^\mu_f(x) J^\nu_f(0)\}\, e^{iq\cdot x} &= i(q^2\eta^{\mu\nu} - q^\mu q^\nu)\, \Pi(q^2+i\varepsilon)\, \mathbb{1} \\ &+ i(\qquad \text{''} \qquad) \times \left[\mathcal{O}\left(\frac{1}{q^{2p}}\right) \right] \mathrm{tr}(F_{\rho\sigma}F^{\rho\sigma})(0) \\ &+ \cdots\cdots \end{aligned}}$$



☆ Is OPE defined at non-perturbative level?

maybe if local operators are defined at non-perturbative level.

☆ The perturbative computation $\Pi_{pert}(q^2)$ for $\Pi(q^2)$

may have zero convergence radius in α_s

still $\left\{ (\text{Borel resummation of } \Pi_{pert}) + \text{minimum non-P contributions to add} \right\} =: [\Pi_{pert}]_B$

may become well-defined.

☆ also quite likely that $\langle\Omega| \mathrm{tr}(F_{\rho\sigma}F^{\rho\sigma})|\Omega\rangle \sim (\Lambda_{QCD})^4 \neq 0$.

This will give rise to (yet another) contribution supp. relatively by $\left(\frac{\Lambda_{QCD}^4}{s^2}\right)$.

from Peskin–Schroeder $\boxed{\frac{\sigma(e^+e^-\to\text{hadron})}{\left[\sigma(e^+e^-\to\mu^+\mu^-)=\frac{4\pi\alpha_e^2}{3s}\right]} =: R(s)}$ data vs theory

$R=R_0\cdot K.$ $\boxed{R_0=\sum_f N_c Q_f^2}$ $\boxed{K\simeq 1+\frac{\alpha_s(\mu)}{\pi}+\cdots}$

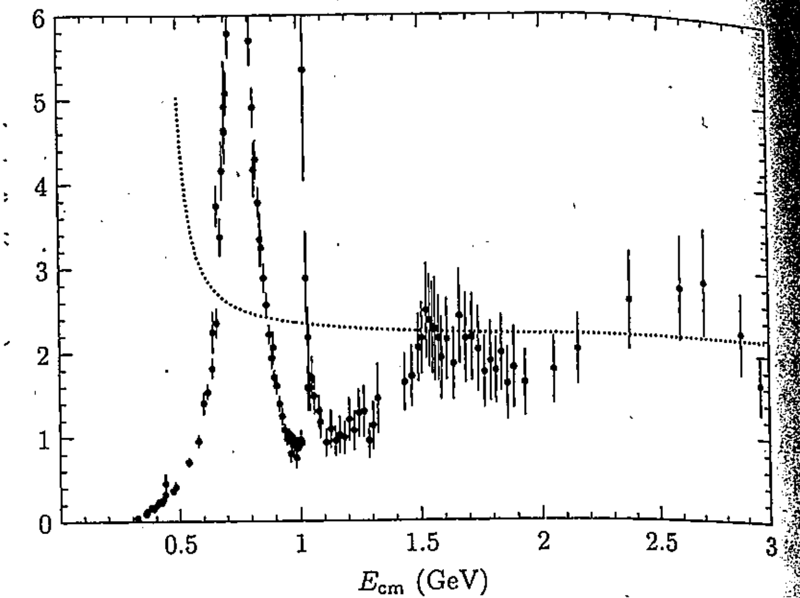
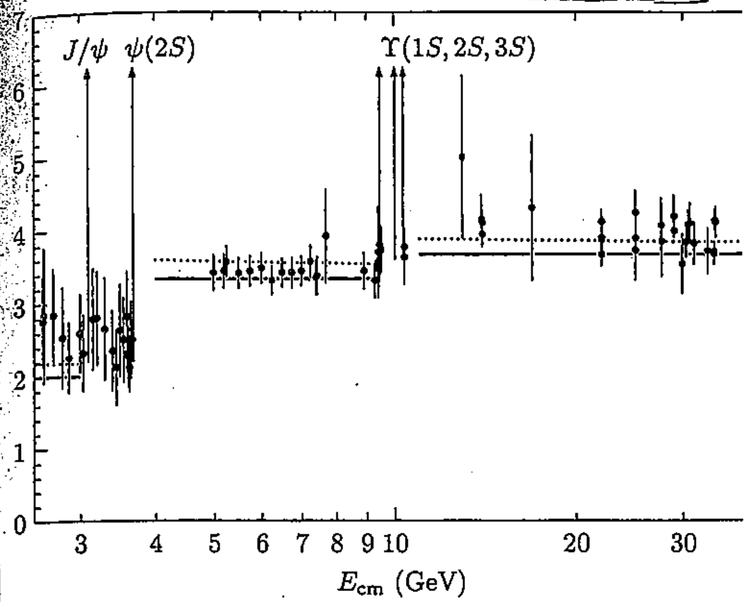


Figure 18.8. Experimental measurements of the total cross section for the reaction $e^+e^-\to$ hadrons at energies below 3 GeV, compared to the prediction of perturbative QCD for 3 quark flavors. The data are taken from the compilation of M. Swartz, *Phys. Rev.* D53, 5268 (1996). Complete references to the various results are given there.

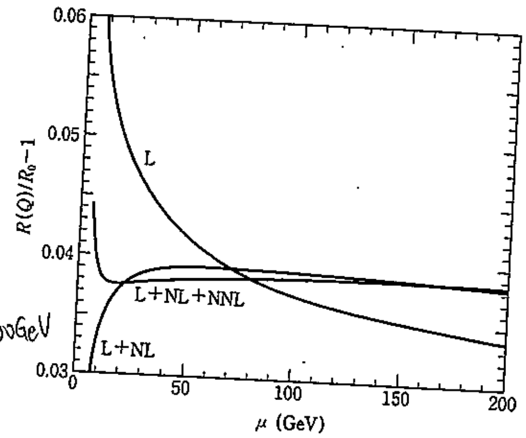
:tions, which reflect the form of the proton wavefunction and are de ed by soft QCD dynamics. However, we saw in Section 17.5 that effects D perturbation theory cause the parton distributions to change their fo . function of the m



igure 5.3. Experimental measurements of the total cross section for action $e^+e^-\to$ hadrons, from the data compilation of M. Swartz, P ev. D53, 5268 (1996). Complete references to the various experiments ven there. The measurements are compared to theoretical predictions f uantum Chromodynamics, as explained in the text. The solid line is mple prediction (5.16).

$e^+e^-\to\mu^+\mu^-$: Helicity Structure

r a reaction is generally easy to c



$\sqrt{s}=100$GeV

By using $\alpha_s(\mu)$ with $\mu\sim\sqrt{s}$ K^L can be a good approximation.

図5.6 $R=\frac{\sigma(e^+e^-\to ハドロン)}{\sigma(e^+e^-\to\mu\bar{\mu}:\text{QED})}$ の QCD 効果のスケール依存性[5]
図には $\frac{R(Q=100\text{GeV})}{R_0}-1$, $R_0=\frac{11}{3}$ を書いてある.
L=対数第1近似 (leading log approximation), NL=第2近似 (next-to-LLA), NNL=第3近似 (next next-to-LLA).

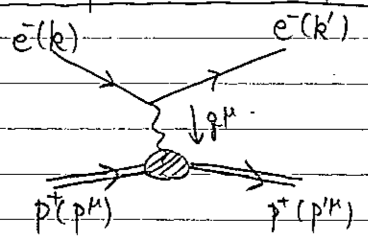
from Nagashima p.139

§ 11 Parton Distribution and Collinear Factorization

§ 11.1 Deep Inelastic Scattering

- experimental foundation for QCD
- an example of IR safe observable in QCD

$e^- + p^+ \longrightarrow e^- + p^+$ elastic scattering



$p'_\mu = (p_\mu + q_\mu)$ but $p^2 = m_p^2 = (p')^2$

$$\Rightarrow W^2 := (p+q)^2 = m_p^2 .$$

$$2p\cdot q + q^2 = 0 \qquad \left(\frac{-q^2}{2p\cdot q} = +1 \right)$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{Mott} \times \underbrace{F(Q^2)}_{\text{form factor of } p^+} \qquad Q^2 := -q^2$$

$e^- + p^+ \longrightarrow e^- + p^+$ inelastic scattering

What happens if $W^2 > m_p^2$? $\gamma^* + p^+ \to (\text{resonance}) \to (\text{anything})$
(incl. p^+ or n)
(#(baryon) conserves)

Ⓐ The resonance peaks dwindle.

$\left[\text{fixed } W, \text{ fixed } \theta . \ (k\cdot p) \nearrow, \ Q^2 \nearrow \quad \frac{Q^2}{2p\cdot q} \nearrow 1 \right]$

Ⓑ The power-law decay with respect to the momentum transfer Q^2 ($e + p \to e + (\text{resonance})$) is not found when $W^2 \gg m_p^2$.
(inelastic region)

$\Longrightarrow$ $\exists$ point-like constituents in a proton?

Ⓒ Bjorken scaling: parametrize

$$\frac{d\sigma(ep \to e + \text{any})}{dx\, dQ^2} = \frac{2\pi\alpha_e^2}{x Q^4} \left[2(1-y)\sigma^{(L)} + \{1 + (1-y)^2\} \tilde{\sigma}^{(T)} \right]$$

$$\Rightarrow \boxed{\sigma^{(L)} \ll \tilde{\sigma}^{(T)}} \text{ and } \boxed{\tilde{\sigma}^{(T)} \approx \text{fn of } (x) \text{ not } Q^2 .}$$

$$\boxed{S = (k+p)^2 \approx 2k\cdot p + m_p^2 , \quad y := \frac{p\cdot q}{p\cdot k} , \quad x := \frac{-q^2}{2p\cdot q}} \Rightarrow \frac{W^2 - m_p^2}{Q^2} = \frac{1}{x} - 1 , \quad \tan^2\left(\frac{\theta}{2}\right) = \frac{m_p^2 Q^2}{\left(4(k\cdot p)(k'\cdot p) + m_p^2 Q^2\right)}$$

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Ⓐ

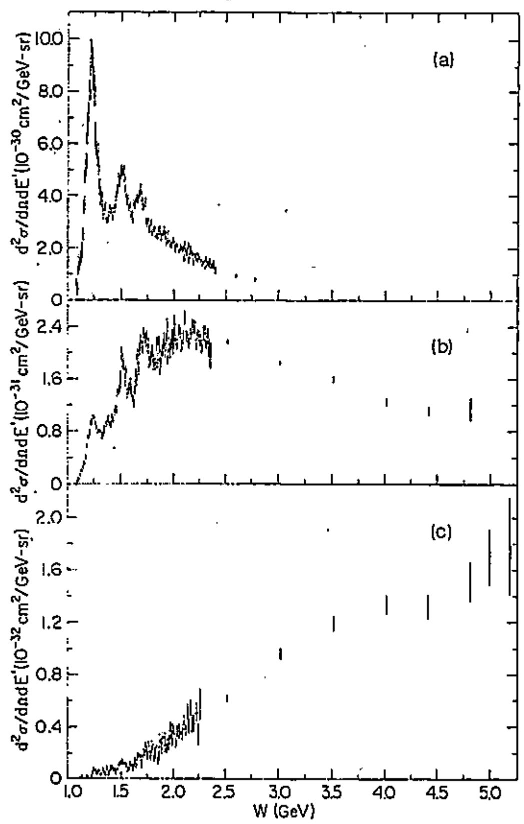


FIG. 2. Three representative radiatively corrected spectra at (a) $\theta=6°$, $E=7$ GeV; (b) $\theta=6°$, $E=16$ GeV, and (c) $\theta=10°$, $E=17.7$ GeV. The ranges of q^2 covered are (a) $0.2 \leq q^2 \leq 0.5$ (GeV/c)^2; (b) $0.7 \leq q^2 \leq 2.6$ (GeV/c)^2; and (c) $1.6 \leq q^2 \leq 7.3$ (GeV/c)^2. The elastic peaks are not shown.

Ⓑ

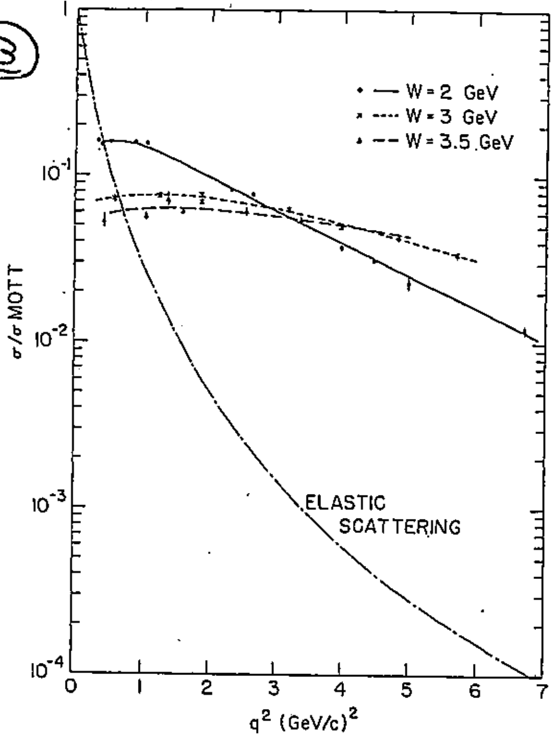
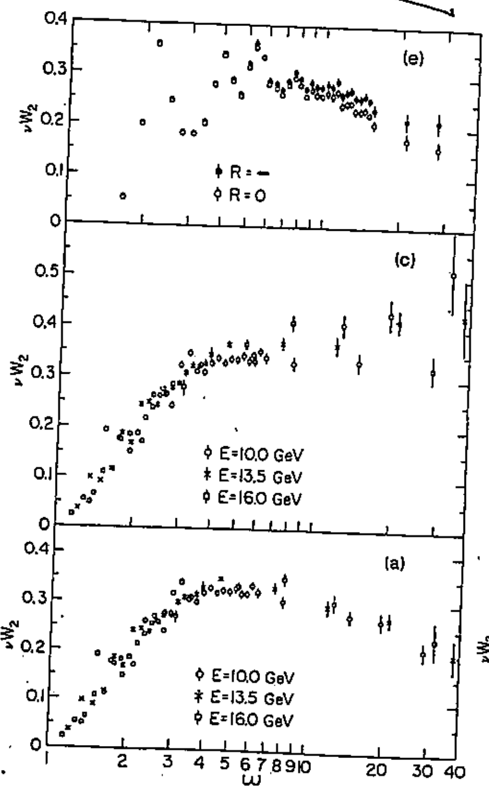


FIG. 1. $(d^2\sigma/d\Omega dE')/\sigma_{\rm Mott}$, in GeV^{-1}, vs q^2 for W = 2, 3, and 3.5 GeV. The lines drawn through the data are meant to guide the eye. Also shown is the cross section for elastic e-p scattering divided by $\sigma_{\rm Mott}$, $(d\sigma/d\Omega)/\sigma_{\rm Mott}$, calculated for $\theta=10°$, using the dipole form factor. The relatively slow variation with q^2 of the inelastic cross section compared with the elastic cross section is clearly shown.

~~re interactions at high energies.~~ Since at
only ~~cross-section measurements~~ at
~~angles~~ are available, we are ~~unable to~~

tal
of
Fi

Ⓒ



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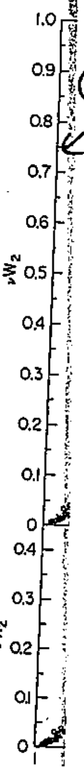
(Fig 2).



νW_2 vs $\omega=2M\nu/q^2$ is shown for various assumptions a
$R=0$. (b) 10° data for $R=0$. (c) 6° data except for 7-Ge
GeV spectrum for $R=0$ and $R=\infty$.

Phys. Rev. D5 ('72) 528.

Ⓒ

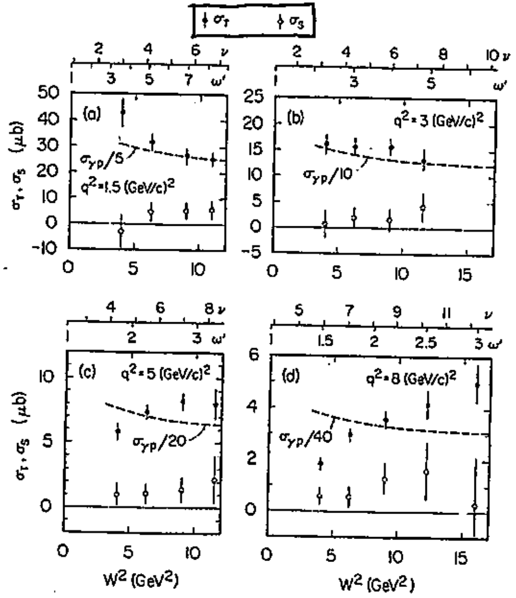
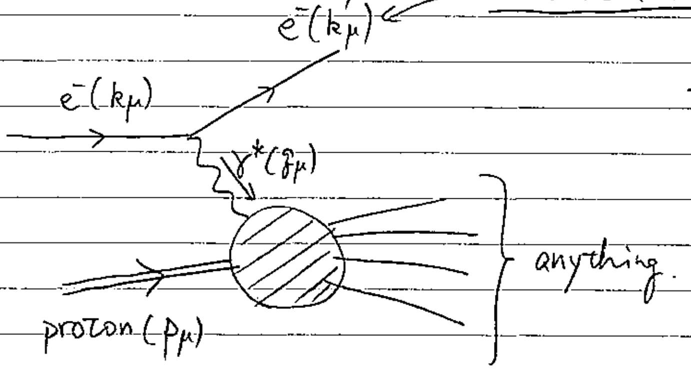


FIG. 8. The values of σ_T and σ_S given in Table III are shown at constant q^2 as a function of W^2 (or ν) for q^2 =1.5, 3, 5, and 8 (GeV/c)^2. Also shown is the ν dependence of the total photoabsorption cross section.

ences ~~would point~~ to a substantial ~~nondiffractive~~ component of the ~~deep-inelastic~~ cross section for ~~values of ω~~ less than approximately six.

APPENDIX: RADIATIVE



$$\underline{q_\mu = (k_\mu - k'_\mu)}$$

$$\Rightarrow \quad q^2 < 0.$$

$\because \cdot (k-q)^2 = (k')^2 = m_e^2 = k^2$

$$\Rightarrow \quad q^2 = \underline{2k\cdot q}.$$

- In the rest frame of $e^-(k_\mu)$,

$k_\mu = (m_e, \vec{0})$

$k'_\mu = (E_e, \vec{k}')$ $\quad E_e > 0.$

$q_\mu = (\underbrace{m_e - E_e}_{\text{negative}}, -\vec{k}')$

$\hookrightarrow$ negative.

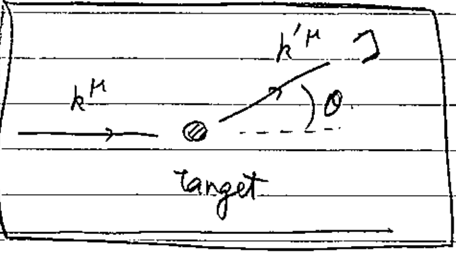
$$\Rightarrow \underline{k\cdot q} < 0$$

In the rest frame of the target proton

$$y = \frac{q\cdot p}{k\cdot p} = \frac{(E-E')}{E}\frac{m_p}{m_p} = \left(\begin{matrix}\text{fraction of}\\ \text{the energy loss}\end{matrix}\right)$$

$$k^\mu = (E, \vec{k}) = (E, k, \vec{0}_T)$$

$$k'^\mu = (E', \vec{k}') = (E', k'\cos\theta, k'\sin\theta)$$



scattering angle θ
(ignore m_e^2 here)

- $Q^2 = (-q^2) = -(E-E')^2 + (E-E'\cos\theta)^2 + (E'\sin\theta)^2$

$$\cong 2EE'(1-\cos\theta)$$

- $1-\cos\theta = \dfrac{2\tan^2(\theta/2)}{1+\tan^2(\theta/2)}$

- $EE' = \dfrac{(m_pE)(m_pE')}{m_p^2} = \dfrac{(k\cdot p)(k'\cdot p)}{m_p^2}$

$$\Rightarrow \quad m_p^2 Q^2 = 4(k\cdot p)(k'\cdot p)\frac{\tan^2(\theta/2)}{1+\tan^2(\theta/2)}$$

$$\boxed{\tan^2(\theta/2) = \frac{m_p^2 Q^2}{4(k\cdot p)(k'\cdot p) + m_p^2 Q^2}}$$

$$\boxed{\frac{W^2 - m_p^2}{(-q^2)} = \frac{(p+q)^2 - m_p^2}{(-q^2)} = \frac{2p\cdot q + q^2}{-q^2} = \left(\frac{1}{x} - 1\right)}$$

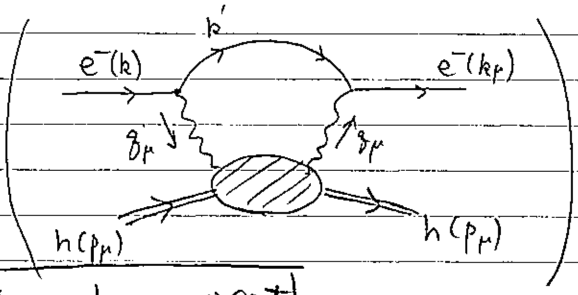
At high energy (where m_p^2 is negligible)

$$\boxed{\underline{s\,y\,x} \simeq (2k\cdot p)\cdot\left(\frac{q\cdot p}{k\cdot p}\right)\cdot\left(\frac{-q^2}{2q\cdot p}\right) = -q^2 = \underline{Q^2}}$$

§11.2. DIS Structure Functions

(Ignore m_e, m_p)

$$\sigma(e^- + p^+ \to e^- + \text{anything}) \simeq \frac{1}{2S}\, 2\,\mathrm{Im}\left(\mathcal{M}(e^-p^+ \to e^-p^+)\right)$$ optical thm



$\boxed{\text{The unknown part}}$

$$\int d^4x \int d^4y\ (-ieQ_f)^2 \langle h(p)|\,T\{J_f^\mu(x)\,J_f^\nu(y)\}\,|h(p)\rangle\, e^{iq'\cdot x}\, e^{-iq\cdot y}.$$

$\|$ translational invariance of the ME.

$$\int d^4x \int d^4y\ (-ieQ_f)^2 \langle h(p)|\,T\{J_f^\mu(0)\ J_f^\nu(y-x)\}\,|h(p)\rangle\, e^{iq'\cdot x}\, e^{-iq\cdot y}.$$

$$= \underbrace{\int d^4x\ e^{i(q'-q)\cdot x}}_{\substack{\| \\ (2\pi)^4\delta^4(q'-q)}} \underbrace{\int d^4(y-x)\ (-ieQ_f)^2 \langle h(p)|\,T\{J_f^\mu(0)\,J_f^\nu(y-x)\}\,|h(p)\rangle\, e^{-iq\cdot(y-x)}}_{\substack{\| \\ i e^2 T^{\mu\nu}(p,q)}}$$ Compton Tensor.

If the e^- beam is not polarized, and the spin of $e^-(k')$ is not measured,

$$\boxed{\begin{aligned} &\sigma(e^-p^+ \to e^- + X) \simeq (\text{homework IX-5}) \simeq \\ &\frac{1}{4p\cdot k}\int dQ^2 \int dy\ \frac{\alpha_e^2}{Q^4}\ 2\left[k_\mu k'_\nu + k'_\mu k_\nu - \eta_{\mu\nu}(k\cdot k')\right]\ 2\,\mathrm{Im}\left(T^{\mu\nu}\right) \\ &= \int dQ^2 \int dx\ \frac{y^2\alpha_e^2}{Q^6}\left[k_\mu k'_\nu + k'_\mu k_\nu - \eta_{\mu\nu}(k\cdot k')\right]\ 2\,\mathrm{Im}\left(T^{\mu\nu}\right). \end{aligned}}$$

From the gauge invariance of QED,

$$q_\mu T^{\mu\nu} = 0 \qquad T^{\mu\nu} q_\nu = 0$$

$$\Rightarrow \boxed{T^{\mu\nu} = (4\pi)\left\{\left(-\eta^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) T_1 + \frac{1}{(p\cdot q)}\left[p^\mu - \frac{(p\cdot q)}{q^2} q^\mu\right]\left[p^\nu - \frac{(p\cdot q)}{q^2} q^\nu\right] T_2\right\}}$$

parametrized by 2 functions.

$$F_1 := 2\,\mathrm{Im}(T_1) \qquad F_2 := 2\,\mathrm{Im}(T_2)$$

Then

$$\boxed{\frac{d\sigma_{DIS}}{dQ^2 dx}} = \frac{y^2 \alpha_e^2}{Q^6}(4\pi)\left[k_\mu k'_\nu + k'_\mu k_\nu - \eta_{\mu\nu}(k\cdot k')\right]\left\{\left(-\eta^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right)F_1 + \frac{1}{p\cdot q}\left[\quad\right]^\mu\left[\quad\right]^\nu F_2\right\}$$

$$= \frac{y^2\alpha_e^2}{Q^6}(4\pi)\left(\underbrace{\left[\begin{array}{l} 2(k\cdot k') + \\ \left\{\frac{2(q\cdot k)(q\cdot k')}{q^2} - (k\cdot k')\right\}\end{array}\right]}_{0} F_1 + \frac{F_2}{(p\cdot q)}\left[\begin{array}{l} 2(p\cdot k)(p\cdot k') \\ -2\frac{p\cdot q}{q^2}\underbrace{\{(p\cdot k)(q\cdot k') + (p\cdot k')(q\cdot k) - (p\cdot q)(k\cdot k')\}}_{0} \\ +\frac{(p\cdot q)^2}{(q^2)^2}\underbrace{\{2(q\cdot k)(q\cdot k') - q^2(k\cdot k')\}}_{0}\end{array}\right]\right)$$

use $0 \simeq (k')^2 \simeq (k-q)^2 \simeq -2k\cdot q + q^2 \Rightarrow k\cdot q \simeq q^2/2$ (Ward-Takahashi id.)

$q^2 = (k-k')^2 \simeq -2k\cdot k' \Rightarrow k\cdot k' \simeq -q^2/2$.

$$= \boxed{\frac{4\pi\alpha_e^2}{Q^4}\left(y^2 F_1 + \frac{(1-y)}{x}F_2\right)}$$

) reorganized

$$= \boxed{\frac{4\pi\alpha_e^2}{Q^4}\left[\frac{(1-y)}{x}(F_2 - 2xF_1) + \{1+(1-y)^2\}F_1\right]}$$

$F_1(x, Q^2)$, $F_2(x, Q^2)$: structure functions.

(experimental data) ⇒
- Bjorken scaling: F_1, F_2 depend primarily on x not on Q^2
  ⟷ point-like constituent.
- Callan-Gross relation: $(F_2 - 2xF_1) \ll F_2$
  ⟷ spin-1/2 constituent (quark)

$$\overline{|\mathcal{M}|^2} \underset{\substack{\uparrow \\ (ef_{1/2} \to ef_{1/2})}}{\propto} \frac{\hat{s}^2 + \hat{u}^2}{\hat{t}^2} \propto \frac{(k\cdot p)^2 + (k'\cdot p)^2}{Q^4} = \frac{(k\cdot p)^2}{Q^4}\{1+(1-y)^2\}$$

← (homework IX-3)

§11.3 Evaluation of $T^{\mu\nu}$ by OPE

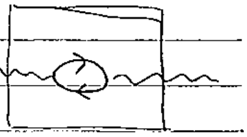
$$T^{\mu\nu} := iQ_f^2 \int d^4y \, \langle h(\vec{p}) | T\{J_f^\mu(0) \, J_f^\nu(y)\} | h(\vec{p}) \rangle \, e^{-iq\cdot y} \quad \text{w/ space-like } q^2<0.$$

$$\left(\text{instead of} \quad \Pi^{\mu\nu} := ie^2Q_f^2 \int d^4y \, \langle \Omega | T\{J_f^\mu(0) \, J_f^\nu(y)\} | \Omega \rangle \, e^{-iq\cdot y} \quad \text{w/ } q^2>0 \right)$$

vacuum polarization.

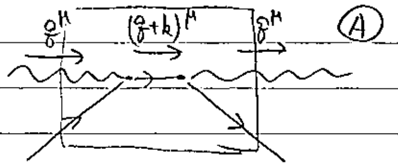
$i\int d^4y \, e^{-iq\cdot y} \, T\{J^\mu(0) J^\nu(y)\}$ OPE

- begins with $(q^2\eta^{\mu\nu} - q^\mu q^\nu)\,\Pi_{\text{ren}}(q^2)\,\mathbb{1}_{\text{operator}}$



But $\operatorname{Im}(\Pi_{\text{ren}}(q^2)) = 0$ if $q^2<0$.

-



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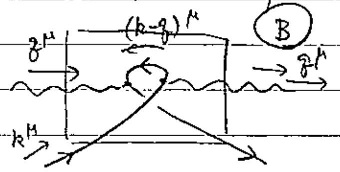


Ⓐ: $i\int d^4y \, e^{-iq\cdot y} \int \frac{d^4k}{(2\pi)^4} \, \bar\psi(-\frac{y}{2}) \, \gamma^\mu \frac{i(\not{q}+\not{k})\, e^{i(k+q)\cdot y}}{(q+k)^2+i\varepsilon} \gamma^\nu \psi(+\frac{y}{2})$

$= (-) \int d^4y \int \frac{d^4k}{(2\pi)^4} \, e^{ik\cdot y} \, e^{\frac{y}{2}\cdot(\overrightarrow{\partial}-\overleftarrow{\partial})} \, \frac{\bar\psi(0)\,\gamma^\mu(\not{q}+\not{k})\gamma^\nu\psi(0)}{(q+k)^2+i\varepsilon}.$

$(2\pi)^4\delta^4\left(k - \frac{i}{2}\underbrace{(\overrightarrow{\partial}-\overleftarrow{\partial})}_{\overleftrightarrow{\partial}}\right)$

$= (-) \, \bar\psi(0) \frac{\gamma^\mu \, i(\not{q}+\frac{i}{2}\overleftrightarrow{\not{\partial}})\,\gamma^\nu}{(q+\frac{i}{2}\overleftrightarrow{\partial})^2+i\varepsilon} \psi(0)$

expand $\left[q^2 + iq\cdot\overleftrightarrow{\partial} - \frac{1}{4}(\overleftrightarrow{\partial})^2\right]$

with respect to $\left(q\cdot\overleftrightarrow{\partial}/q^2\right)$

(equation of motion

$\not{D}\psi=0 \Rightarrow \not{D}\not{D}\psi = \left(D^2 + \frac{ig}{4}[\gamma^\mu,\gamma^\nu]F_{\mu\nu}\right)\psi = 0.$

Ⓑ: $i\int d^4y \, e^{-iq\cdot y} \int \frac{d^4k}{(2\pi)^4} \, \bar\psi(+\frac{y}{2}) \, \gamma^\nu \frac{i(\not{k}-\not{q})\, e^{-i(k-q)\cdot y}}{(k-q)^2+i\varepsilon} \gamma^\mu \psi(-\frac{y}{2}).$

$= \cdots = (+) \, \bar\psi(0) \frac{\gamma^\nu(\not{q}-\frac{i}{2}\overleftrightarrow{\not{\partial}})\,\gamma^\mu}{(q-\frac{i}{2}\overleftrightarrow{\partial})^2+i\varepsilon} \psi(0).$

OPE $i\int d^4y\ e^{-iq\cdot y}\, T\{J_f^{\mu}(-y/2)\, J_f^{\nu}(+y/2)\}$.

$$= \left(q^2\eta^{\mu\nu} - q^{\mu}q^{\nu}\right)\Pi_{\text{ren.}}(q^2)\ \mathbb{1}$$

$$+ \sum_{j=1}^{\infty} C^{\mu\nu}_{\lambda_1\cdots\lambda_j}(q)\ \underbrace{\left[\bar{\psi}_f\gamma^{\lambda_1}\left(\tfrac{i}{2}\overleftrightarrow{\partial}\right)^{\lambda_2}\cdots\left(\tfrac{i}{2}\overleftrightarrow{\partial}\right)^{\lambda_j}\psi_f\right](0)}_{\text{local operator}} \longrightarrow \boxed{\text{twist} = (2+j)-j = 2}$$

$$+ \left(\mu\leftrightarrow\nu \text{ antisymmetric part.}\right)$$

$$+ (\text{coeff.})\times \text{operator such as } \left(\bar{\psi}_f\gamma^{\kappa}F^{\rho\sigma}\psi_f\right)(0)\ \cdots\ \underline{\text{twis} = 5-1=4}$$

$$(\text{twist}) := (\text{naive operator dim}) - \text{spin}\left(\text{repr. of } SO(3,1)\right)$$

Insert those local operators in $\langle h(\vec{p})|\qquad|h(\vec{p})\rangle$.

$$\langle h(\vec{p})|\underbrace{\left[\bar{\psi}\gamma^{\lambda_1}\left(\tfrac{i}{2}\overleftrightarrow{D}\right)^{\lambda_2}\cdots\left(\tfrac{i}{2}\overleftrightarrow{D}\right)^{\lambda_j}\psi\right]}_{\text{sym. traceless}}|h(\vec{p})\rangle =: p^{\lambda_1}p^{\lambda_2}\cdots p^{\lambda_j}\underline{A_j}$$

non-perturbative information

Use the explicit expressions of $C^{\mu\nu}_{\lambda_1\cdots\lambda_j}(q)$

to obtain (homework IX-4)

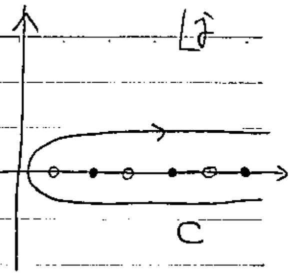
$$\boxed{\left(\underset{\substack{\mu\nu\text{-sym,}\\ \text{twist-2}}}{T^{\mu\nu}}\right) = \left\{\left(-\eta^{\mu\nu}+\frac{q^{\mu}q^{\nu}}{q^2}\right) + \frac{1}{(p\cdot q)}\left[p^{\mu}-q^{\mu}\frac{p\cdot q}{q^2}\right]\left[p^{\nu}-q^{\nu}\frac{(p\cdot q)}{q^2}\right]2x\right\} \times\sum_{j=1}^{\infty}\left[1+(-)^j\right]\left(\frac{1}{x}\right)^j\left(+\frac{A_j}{2}\right)\left(Q_f^2\right)}$$

→ consistent with the Ward-Takahashi identity

→ Callan-Gross relation. ($F_2 = 2xF_1$)

$$T_1 = \frac{+1}{8\pi} \sum_{j=1}^{\infty} [1+(-)^j] \frac{1}{x^j} A_j Q_f^2 \Leftarrow$$ expansion @ $1 \ll x$

$$= \frac{1}{8\pi} \int_C \frac{dj}{2i} \left[-\frac{1+e^{-\pi i j}}{\sin(\pi j)} \right] \frac{1}{x^j} A_f^+(j) Q_f^2$$ continuation to $x<1$.



- $A^{(+)}(j)$: holomorphic fcn on j. $A^+(j \in 2\mathbb{N}) = A_{j \in 2\mathbb{N}}$.
- $\frac{1+e^{-\pi i j}}{\sin(\pi j)}$: pole @ $j \in 2\mathbb{Z}$ residue $= \frac{2}{\pi}$.

$$\boxed{2\,\mathrm{Im}(T_1) = \frac{1}{4\pi} \int_{-i\infty}^{+i\infty} \frac{dj}{2i} \frac{1}{x^j} A^+(j)\, Q_f^2}$$

$$\boxed{\int_0^{+\infty} dx\, [2\,\mathrm{Im}(T_1)](x)\, x^{j-1} = \frac{1}{4} A^+(j)\, Q_f^2 .}$$

Mellin transformation $\int_0^{+\infty} dx\, f(x)\, x^{j-1} =: \hat{f}(j)$.

inverse Mellin transformation $f(x) = \int_{-i\infty}^{+i\infty} \frac{dj}{2\pi i} \left(\frac{1}{x}\right)^j \hat{f}(j)$.

(Fourier transformation $\ln(x) \leftrightarrow j/\sqrt{-1}$.)

(Laplace transformation $\ln(x) \leftrightarrow j$)

Structure functions are given by the inverse Mellin transform of the proton matrix elements of twist-2 quark operators.