OPE $\quad i\int d^4y\; e^{-iq\cdot y}\, T\left\{ J_f^{\mu}(-y/2)\, J_f^{\nu}(+y/2)\right\}$.

$$= \left(q^2\eta^{\mu\nu} - q^{\mu}q^{\nu}\right)\Pi_{\mathrm{ren}}(q^2)\,\mathbb{1}$$

local operator

$$+\sum_{j=1}^{\infty} C^{\mu\nu}_{\lambda_1\cdots\lambda_j}(q)\left[\bar\psi_f\gamma^{\lambda_1}\left(\tfrac{i}{2}\overleftrightarrow{\partial}\right)^{\lambda_2}\cdots\left(\tfrac{i}{2}\overleftrightarrow{\partial}\right)^{\lambda_j}\psi_f\right](0) \longrightarrow \boxed{\text{twist} = (2+j)-j = 2}$$

$$+\left(\mu\leftrightarrow\nu \text{ antisymmetric part.}\right)$$

$$+\left(\text{coeff.}\right)\times \text{operator such as } \left(\bar\psi_f\gamma^{\kappa}F^{\rho\sigma}\psi_f\right)(0) \;\cdots\; \text{twis} = 5-1=4$$

$$(\text{twist}) := (\text{naive operator dim}) - \text{spin}\left(\text{repr. of } SO(3,1)\right)$$

Insert those local operators in $\langle h(\vec p)|\qquad |h(\vec p)\rangle$.

$$\langle h(\vec p)|\underbrace{\left[\bar\psi\gamma^{\lambda_1}\left(\tfrac{i}{2}\overleftrightarrow{D}\right)^{\lambda_2}\cdots\left(\tfrac{i}{2}\overleftrightarrow{D}\right)^{\lambda_j}\psi\right]}_{\text{sym. traceless}}|h(\vec p)\rangle =: p^{\lambda_1}p^{\lambda_2}\cdots p^{\lambda_j}\underline{A_j}$$

non-perturbative information

Use the explicit expressions of $C^{\mu\nu}_{\lambda_1\cdots\lambda_j}(q)$

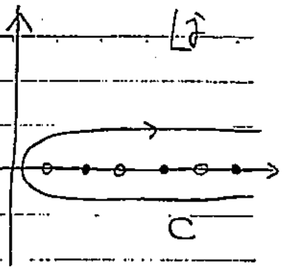
to obtain (homework IX-4)

$$\left(T^{\mu\nu}\right)_{\substack{\mu\nu\text{-sym,}\\ \text{twist-2}}} = \left\{\left(-\eta^{\mu\nu}+\frac{q^{\mu}q^{\nu}}{q^2}\right) + \frac{1}{(p\cdot q)}\left[p^{\mu}-q^{\mu}\frac{p\cdot q}{q^2}\right]\left[p^{\nu}-q^{\nu}\frac{(p\cdot q)}{q^2}\right]2x\right\}$$

$$\times\sum_{j=1}^{\infty}\left[1+(-)^j\right]\left(\frac{1}{x}\right)^j\left(+\frac{A_j}{2}\right)\left(Q_f^2\right)$$

→ consistent with the Ward–Takahashi identity

→ Callan–Gross relation. $\left(F_2 = 2xF_1\right)$



$q = u.\bar{u}.d\bar{d}$

$$T_1 = \frac{+1}{8\pi}\sum_{j=1}^{\infty}[1+(-)^j]\frac{1}{x^j}\sum_f A_j^f Q_f^2 \Leftarrow$$ expansion @ $1 \ll x$

$$= \frac{1}{8\pi}\int_C \frac{dj}{2i}\left[\frac{1+e^{-\pi i j}}{\sin(\pi j)}\right]\frac{1}{x^j}\sum_f A_f^+(j) Q_f^2$$ continuation to $x<1$.

- $A^{+}(j)$: holomorphic fcn on j $A^+(j \in 2\mathbb{N}) = A_{j\in 2\mathbb{N}}$.
- $\frac{1+e^{-\pi i j}}{\sin(\pi j)}$: pole @ $j \in 2\mathbb{Z}$ residue $= \frac{2}{\pi}$.

$$\boxed{2\,\mathrm{Im}(T_1) = \frac{1}{4\pi}\int_{-i\infty}^{+i\infty}\frac{dj}{2i}\,\frac{1}{x^j}\,A_f^+(j)\,Q_f^2}$$

$$\boxed{\int_0^{+\infty} dx\,\left[2\,\mathrm{Im}(T_1)\right](x)\,x^{j-1} = \frac{1}{4}A_f^+(j)\,Q_f^2.}$$

Mellin transformation $\int_0^{+\infty} dx\, f(x)\, x^{j-1} =: \hat{f}(j)$

inverse Mellin transformation $f(x) = \int_{-i\infty}^{+i\infty}\frac{dj}{2\pi i}\left(\frac{1}{x}\right)^j \hat{f}(j)$.

(Fourier transformation $\ln(x) \leftrightarrow -j/\sqrt{-1}$.)

(Laplace transformation $\ln(x) \leftrightarrow j$)

Structure functions are given by the inverse Mellin transform of the proton matrix elements of twist-2 quark operators.

§11.4 Parton Distribution Function (PDF)

$$\boxed{f_q(x) := \frac{1}{4\pi}\int_{-\infty}^{+\infty} d\kappa\ e^{i\kappa x} \langle h(\vec{p})| \left[\bar{\psi}_q\left(-\frac{\bar{n}}{2}\kappa\right) \bar{\not{n}}\, \psi_q\left(+\frac{\bar{n}}{2}\kappa\right)\right] |h(\vec{p})\rangle}$$

quark PDF

$$f_{\bar{q}}(x) := \frac{1}{4\pi}\int_{-\infty}^{+\infty} d\kappa\ e^{i\kappa x} \langle h(\vec{p})| \left[\left(\overline{\psi_q^c}\right)\left(-\frac{\bar{n}}{2}\kappa\right) \bar{\not{n}} \left(\psi_q^c\right)\left(+\frac{\bar{n}}{2}\kappa\right)\right] |h(\vec{p})\rangle$$

$$= \frac{-1}{4\pi}\int_{-\infty}^{+\infty} d\kappa\ e^{i\kappa x} \langle h(\vec{p})| \left[\bar{\psi}_q\left(+\frac{\bar{n}}{2}\kappa\right) \bar{\not{n}}\, \psi_q\left(-\frac{\bar{n}}{2}\kappa\right)\right] |h(\vec{p})\rangle$$

$$= \frac{-1}{4\pi}\int_{-\infty}^{+\infty} d\kappa'\ e^{-i\kappa' x} \langle h(\vec{p})| \left[\bar{\psi}_q\left(-\frac{\bar{n}}{2}\kappa'\right) \bar{\not{n}}\, \psi_q\left(+\frac{\bar{n}}{2}\kappa'\right)\right] |h(\vec{p})\rangle = -f_q(-x)$$

anti-quark PDF

$$\bar{n}^\mu = \frac{\bar{p}^\mu}{(p\cdot q)} \qquad (\bar{p}^\mu : \text{light like vector with } p\cdot\bar{p} = p\cdot q)$$

κ : dimensionless parameter for integration

- (Wilson line, gauge-invariance etc. : we will come back later)
- For a quark with momentum k^μ collinear to P^μ,

$$e^{\pm i\frac{\bar{n}}{2}\kappa\cdot k} \sim e^{\pm i\frac{\bar{n}'}{2}\kappa\cdot k} \quad \text{so long as } \bar{n}^- = (\bar{n}')^-.$$

So, it is OK to use $\bar{n}'^\mu := \frac{q^\mu}{(p\cdot q)}$ instead of $\bar{n}^\mu$.

For even j,

$$\int_0^{+\infty} dx\, x^{j-1}\{f_q(x) + f_{\bar{q}}(x)\} = \frac{1}{2}\int_{-\infty}^{+\infty} dx\, x^{j-1}\{f_q(x) + f_{\bar{q}}(x)\}$$

$$= \int_{-\infty}^{+\infty}\frac{dx}{2}\int_{-\infty}^{+\infty}\frac{d\kappa}{4\pi}\left[(-i\partial_\kappa)^{j-1} e^{i\kappa x}\right]\left(\langle\vec{p}|\left[\bar{\psi}\left(-\frac{\bar{n}}{2}\kappa\right)\bar{\not{n}}\,\psi\left(+\frac{\bar{n}}{2}\kappa\right)\right]|\vec{p}\rangle - \langle\vec{p}|\left[\bar{\psi}\left(+\frac{\bar{n}}{2}\kappa\right)\bar{\not{n}}\,\psi\left(-\frac{\bar{n}}{2}\kappa\right)\right]|\vec{p}\rangle\right)$$

$$= \frac{1}{4}\left[(i\partial_\kappa)^{j-1}\left(\langle\vec{p}|[-\cdots+]|\vec{p}\rangle - \langle\vec{p}|[+\cdots-]|\vec{p}\rangle\right)\right]\Bigg|_{\kappa=0}$$

$$= \frac{1}{4}\langle\vec{p}|\left[\bar{\psi}\,\bar{\not{n}}\left(\frac{\bar{n}}{2}\cdot i\overleftrightarrow{\partial}\right)^{j-1}\psi\right]|\vec{p}\rangle\left[1-(-)^{j-1}\right]$$

$$= \frac{1}{2}A_j. \qquad (\bar{n}\cdot p = 1)$$

So,
$$\boxed{\begin{aligned} F_1(x) &\overset{\text{tree}}{=} \int_{-i\infty}^{+i\infty} \frac{dj}{2\pi i} \frac{1}{x^j} \frac{1}{4}\left(\sum_f Q_f^2 A_f^+(j)\right) \\ \left\{f_q(x)+f_{\bar{q}}(x)\right\} &= \int_{-i\infty}^{+i\infty} \frac{dj}{2\pi i} \frac{1}{x^j} \frac{1}{2} A_f^+(j) \end{aligned}}$$

$(f = u, d. s, \cdots)$

$$\Rightarrow \boxed{F_1(x) \overset{\text{tree}}{=} \frac{1}{2} \sum_f Q_f^2 \left\{f_q(x) + f_{\bar{q}}(x)\right\}}$$

At $\mathcal{O}(\alpha_s)$ gluon PDF also contributes to the DIS structure functions

(The Callan-Gross relation $F_2 = 2x F_1$ no longer holds.)

$\boxed{\text{parton model.}}$ (idea)

replace $\langle h(\vec p)| \cdots |h(\vec p)\rangle$

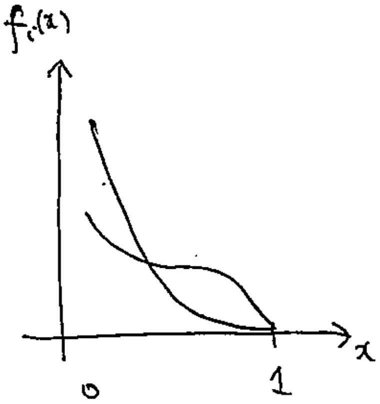
by $\sum_i \int_0^1 \frac{dz}{z} f_{q_i}(z) \; \langle q_i(z\vec p)| \cdots |q_i(z\vec p)\rangle$

$q_i = u, d, \cdots$
$\quad\;\; u^c, d^c \cdots$

$\Longrightarrow$ direct computation of $2\,\mathrm{Im}\,T_1 \equiv F_1$, $2\,\mathrm{Im}\,T_2 \equiv F_2$

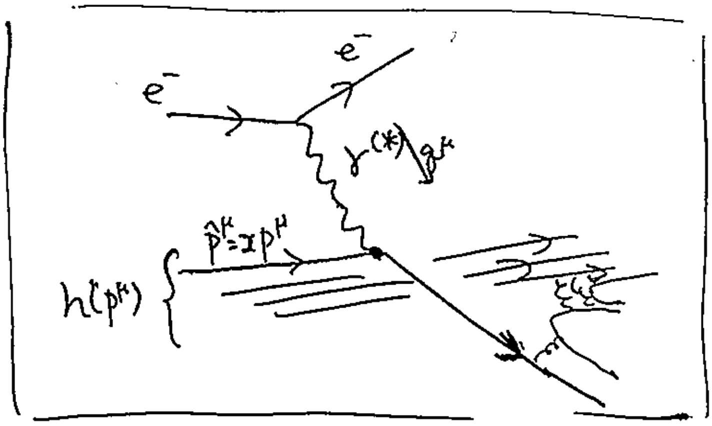
$\rightarrow \; F_1 = \frac{1}{2} Q_q^2 \left[f_q(x) + f_{\bar q}(x) \right]$

(homework).



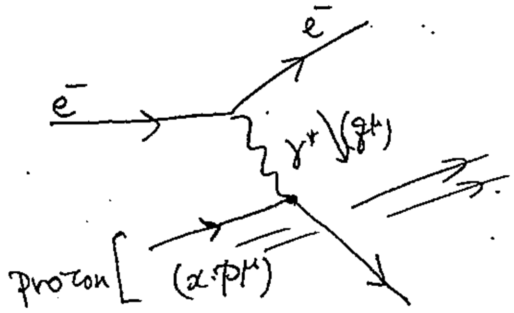
hadron : parton
w/ longitudinal momentum fraction. x.
$f_{q_i}(x)$

other partons : remain unimportant.
if $(-q^2) \gg \Lambda_{QCD}^2$

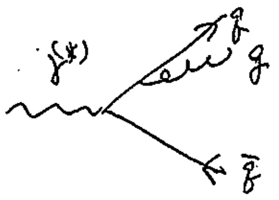


§ 11.5 Initial State Radiation (ISR)

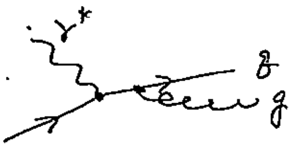
(cf. FSR)



$\boxed{\text{DIS}}$



(s-channel)

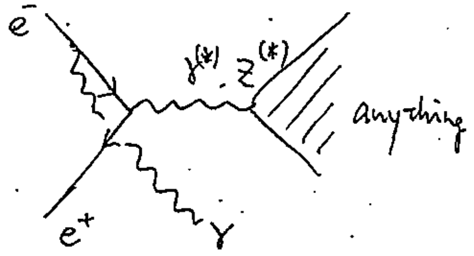


(t-channel)

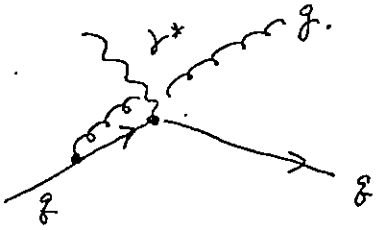
FSR
↓
soft γ collinear divergence.

ISR

(s-channel)



(t-channel)



- ☆ divergence for the same reason as in FSR.
- ☆ un-observable (as in FSR) because the radiation goes down the beam pipe.

Does divergence cancel in observables ? ~~~in DIS~~~

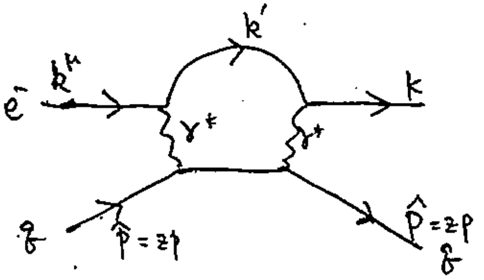
Does divergence cancel in observables in DIS?

QED: see Peskin–Schroder. § 6.5

QCD: ? confinement. hadron...

§ 11.6 DGLAP equation.

< Dokshitzer – Gribov – Lipatov – Altarelli – Parisi >



$$(2\,\mathrm{Im}\,\mathcal{M}) = \int d\Pi\, (\mathcal{M}^\dagger \mathcal{M})$$

unitarity.

(total inelastic cross sectio

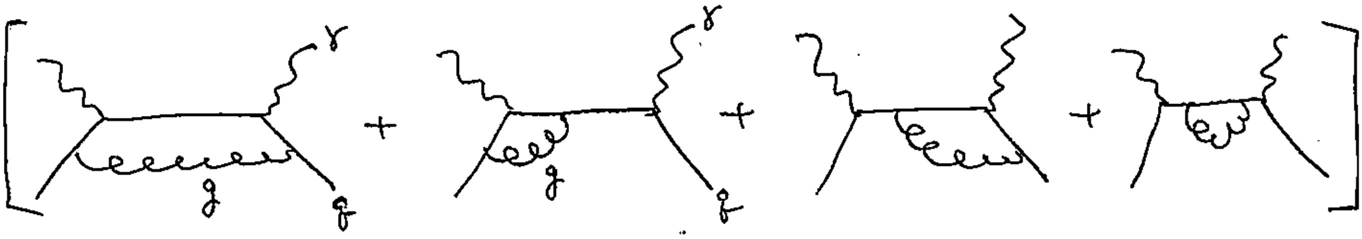
$\swarrow$

Im [forward amplitude.].

Higher-order loop correction?

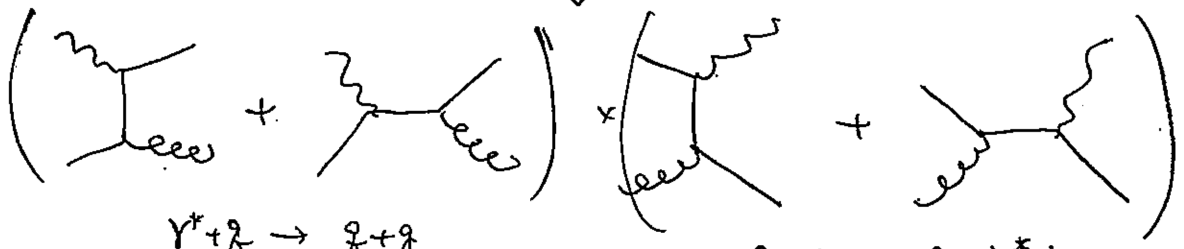
< ~~ignore~~ omit. lepton line. >

$[\gamma^* + q \to \gamma^* + q]$



$\Downarrow$

real gluon emission.



$\gamma^* + q \to q + g$

$q + g \to q + \gamma^*$.

$$\Delta(2\,\mathrm{Im}\,T^{\mu\nu}) = \underbrace{\int_0^1 \frac{dz}{z}\, f_q(z)}_{\text{parton model.}} \times \Big[(-) \times \big[i\mathcal{M}^{\mu\nu}(\gamma^* + q \to \gamma^* + q) \big]^{(\text{spin-average})}$$

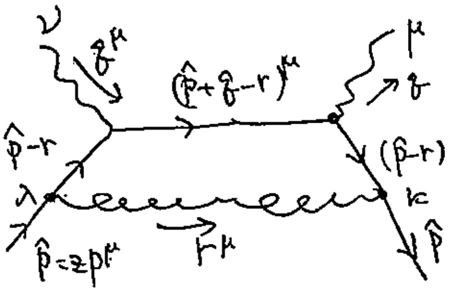
with on-shell propagators $\boxed{\frac{i}{p^2 - m^2 + i\varepsilon} \Rightarrow 2\pi\delta(p^2 - m}$ $\Big]$

Cutkosky rule

Peskin–Schröder §7.3.

eg. <u>1st graph.</u>

$$\Delta^{(1)}(2\mathrm{Im}\,T^{\mu\nu}) = \int_0^1 \frac{dz}{z} f_q(z) \times (-)\int \frac{1}{2}\sum_r \bar{u}_r(\hat{p})\Big[(-igt^a\gamma^\kappa)\frac{i(\hat{\not{p}}-\not{r})}{(\hat{p}-r)^2+i\varepsilon}(iQ_q\gamma^\mu)\frac{i(\hat{\not{p}}+\not{q}-\not{r})}{(\hat{p}+q-r)^2+i\varepsilon}$$



$$(iQ_q\gamma^\nu)\frac{i(\hat{\not{p}}-\not{r})}{(\hat{p}-r)^2+i\varepsilon}(-igt^b\gamma^\lambda)\Big]u_r(\hat{p})$$

$$\times\left(\frac{-i\,\eta_{\kappa\lambda}\,\delta_{ab}}{r^2+i\varepsilon}\right)\frac{d^4r}{(2\pi)^4}$$

with
$$\begin{cases} \dfrac{i}{(\hat{p}+q-r)^2+i\varepsilon} \Rightarrow (2\pi)\,\delta((\hat{p}+q-r)^2) \\[2ex] \dfrac{i}{r^2+i\varepsilon} \Rightarrow (2\pi)\,\delta(r^2). \end{cases}$$

spin average...
$$\mathrm{Tr}\Big[\sum_r\big(u_r(\hat{p})\,\bar{u}_r(\hat{p})\big)\ \text{----}\Big] = \mathrm{Tr}\big[(\hat{\not{p}})\ \text{----}\big].$$

Feynman gauge.

- - - - - - - - - - - - -

If we were to use the tree level result.

$$\eta_{\mu\nu}(2\mathrm{Im}\,T^{\mu\nu}) = \frac{2\pi}{x}(F_2 - 6xF_1) \longrightarrow -(4\pi)\,Q_q^2\big[f_q(x)+f_{\bar{q}}(x)\big]$$

$$\Rightarrow \boxed{-\frac{1}{4\pi}\eta_{\mu\nu}\,\Delta(2\mathrm{Im}\,T^{\mu\nu})} \qquad \text{how is it like?}$$

$$\int\frac{d^4r}{(2\pi)^4}(2\pi)^2\delta(r^2)\,\delta((\hat{p}+q-r)^2) \Rightarrow$$

set an axis.
$$\begin{cases} \hat{p} = z(p.p.\vec{0}) \\ q^\mu = (Q^0, Q^3, \vec{0}) \end{cases}$$

$$r^\mu = (r^0, r^3, \vec{r}_T).$$

- $|\vec{r}_T|$ set by $\delta(r^2)$
- focus on a region. $r^\mu \not\parallel p^\mu$ $(r^0 \sim r^3) \Leftrightarrow (r^+, r^-, \vec{r}_T) \sim (1, \lambda^2, \lambda)$.

$\Rightarrow (\hat{p}+q-r)^2 \simeq \chi z(2p\cdot q) + q^2$ if $(\Delta r)^\mu$ = small.

$r^\mu = \vec{r}_T + z\underset{(1-\chi)}{\cancel{x}}p^\mu + (\Delta r)^\mu$ $\Rightarrow [z\chi - x \simeq 0$

$$\boxed{-\frac{1}{4\pi}\left(2\,\mathrm{Im}\,T^{\mu\nu}\right)\eta_{\mu\nu}} \simeq Q_q^2 f_q(x) +$$

$$Q_q^2 \int \frac{dz}{z} f_q(z) \frac{\left[-g^2 C_2(R)\right]}{(-4\pi)} \int \frac{d(p\cdot r)}{(p\cdot q)} \frac{1}{8\pi} \times \left\{ 4\,\overset{①}{\frac{r\cdot q}{\underset{\sim\sim}{(\hat{p}\cdot r)}}} + 8\,\overset{②+③}{\frac{(\hat{p}\cdot q)\ (\hat{p}-r)\cdot q}{\underset{\sim\sim}{(\hat{p}\cdot r)}(2\hat{p}\cdot q+q^2)} \times 2} \right\}$$

apart from $\frac{1}{(\hat{p}\cdot r)}$. use $r_\mu = (1-x)\hat{p}_\mu$.

$$\frac{(\hat{p}\cdot q)}{(\hat{p}\cdot r)} \times 4\left\{(1-x) + \frac{2x}{(1-x)}\right\} = \left\{\frac{1+x^2}{1-x}\right.$$

$$\Rightarrow \left(Q_q^2\right)\text{“}\left[f_q(x)\right]\text{”} \cong Q_q^2 f_q(x)$$

$$\boxed{+\,Q_q^2\,\frac{\alpha_s}{2\pi} C_2(R) \int \frac{d(p\cdot r)}{(p\cdot r)} \int_x^1 \frac{dz}{z} f_q(z) \left\{\frac{1+(x/z)^2}{1-(x/z)}\right\}}$$

⇓

pure logarithmic divergence.
from a region w/ small $(p\cdot r)$.

Just like renormalization

$$\frac{4\pi}{g^2(p^2)} = \frac{4\pi}{g_0^2} + \frac{b}{2\pi}\ln\left(\frac{p^2}{\Lambda^2}\right) \Rightarrow \left[\frac{4\pi}{g_0^2} + \frac{b}{2\pi}\ln\left(\frac{\mu_R^2}{\Lambda^2}\right)\right] + \frac{b}{2\pi}\ln\left(\frac{p^2}{\mu_R^2}\right)$$

$$\left[\ldots\right] \equiv \left(\frac{4\pi}{g^2(\mu_R)}\right)$$

⇑ physical. observable parameter

↑ renormalized coupling.

$$\boxed{\text{“}f_q(x;Q^2)\text{”} = f_q(x;\mu_F^2) + \frac{\alpha_s}{2\pi} C_2(R) \int_{\mu_F^2}^{Q^2} \frac{d(p\cdot r)}{(p\cdot r)} \int_x^1 \frac{dz}{z} f_q(z;\mu_F^2) \frac{1+(x/z)^2}{1-(x/z)}}$$

⇑ observable ⇑ “renormalized”

μ_F: factorization scale.

$(-2\hat{p}\cdot r) \sim (\hat{p}-r)^2$: virtuality of parton q.

radiative corr. with $\boxed{\text{virtuality} \lesssim \mu_F^2}$: swept under the carpet. $f_q(z;\mu_F$

near light cone.

- Wilson's interpretation of renormalization:

  momenta above μ_R : renormalized into $g(\mu_R)$

- (fluctuation / distribution) close to the light cone than μ_F^2 :

  swept into / taken into account in $f_q(x;\mu_F^2)$

$$\boxed{\frac{\partial f_q(x;\mu_F^2)}{\partial \ln(\mu_F^2)} = \frac{\alpha_s}{2\pi} C_2(R) \int_x^1 \frac{dz}{z} \frac{1+(x/z)^2}{1-(x/z)} f_q(z;\mu_F^2)}$$

DGLAP eq

[weakly depe… on μ_F]

$$\underbrace{\frac{1+(x/z)^2}{1-(x/z)}}_{\parallel} \quad P(x/z) \text{ splitting function.}$$

~~to~~

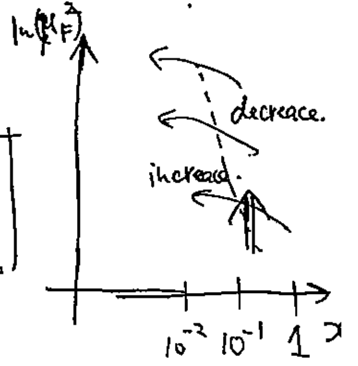
convolution form.

< * extra contribution at $\delta(1-x/z)$ >.

Mellin transform.

$$\boxed{\frac{\partial \tilde{f}_q(j;\mu_F^2)}{\partial \ln(\mu_F^2)} = \frac{\alpha_s}{2\pi} C_2(R)\, \gamma(j)\, \tilde{f}_q(j;\mu_F^2)}$$

$$\boxed{\tilde{f}_q(j) + \tilde{f}_{\bar{q}}(j) = \frac{1}{2} A_j}$$



OPE

operator M.E.

$$\int T\{J^\mu(x) J^\nu(y)\} e^{+iq\cdot(x-y)} d^4y \Rightarrow \sum_j C_j(q^2;\mu_R^2) \left[\bar{\psi}\gamma^\mu (\not{D})^{j-1}\psi\right]_{\mu_R^2}$$

$$\Downarrow$$

$$\sum_j C_j(q^2;\mu_R^2) \langle h | [\quad]_{\mu_R^2} | h \rangle.$$

take care of UV DOF first

"IR" DOF later.

Parton model

$$dz\, f_q(z;\mu_F^2) \quad \cancel{\rightarrow}$$

take care of collinear DOF first

$$\Downarrow$$

$$dz\, f_q(z;\mu_F^2)\, \cancel{P(x/z)}\, \hat{\sigma}(z)$$

hard scatter later.

the same thing.

Factorization into (hard part) / (non-perturbative part

OPE makes it clear in the case of DIS.