- Wilson's interpretation of renormalization:

  momenta above μ_R : renormalized into $g(\mu_R)$

- (fluctuation / distribution) close to the light cone than μ_F^2 :

  swept into / taken into account in $f_q(x;\mu_F^2)$

$$\boxed{\frac{\partial f_q(x;\mu_F^2)}{\partial \ln(\mu_F^2)} = \frac{\alpha_s}{2\pi} C_2(R) \int_x^1 \frac{dz}{z} \underbrace{\frac{1+(x/z)^2}{1-(x/z)}}_{P(x/z)} f_q(z;\mu_F^2)}$$

DGLAP eq

weakly depend on μ_F

$P(x/z)$ splitting function.

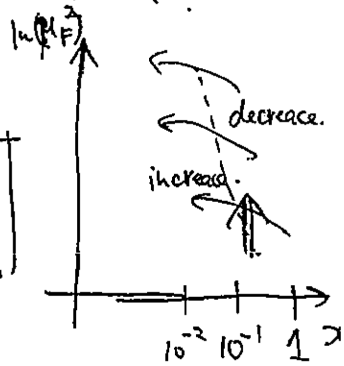
convolution form.

< * extra contribution at $\delta(1-x/z)$ >

Mellin transform.

$$\boxed{\frac{\partial \tilde{f}_q(j;\mu_F^2)}{\partial \ln(\mu_F^2)} = \frac{\alpha_s}{2\pi} C_2(R)\, \gamma(j)\, \tilde{f}_q(j;\mu_F^2)}$$

$$\boxed{\tilde{f}_q(j) + \tilde{f}_{\bar{q}}(j) = \frac{1}{2} A_j}$$



OPE

operator M.E.

$$\int T\{J^\mu(x) J^\nu(y)\} e^{+iq\cdot(x-y)} d^4y \Rightarrow \sum_j C_j(q^2;\mu_R^2) \left[\bar{\psi}\gamma^\mu (\not{D})^{j-1}\psi\right]_{\mu_R^2}$$

$$\Downarrow$$

$$\sum_j C_j(q^2;\mu_R^2) \langle h | [\quad]_{\mu_R^2} | h \rangle .$$

take care of UV DOF first

"IR" DOF later.

Parton model

$dz\, f_q(z;\mu_F^2)$

take care of collinear DOF first

$\Downarrow$

$dz\, f_q(z;\mu_F^2)\, \cancel{P(x/z)}\, \hat{\sigma}(z)$

hard scatter later.

the same thing.

Factorization into (hard part) × (non-perturbative part)

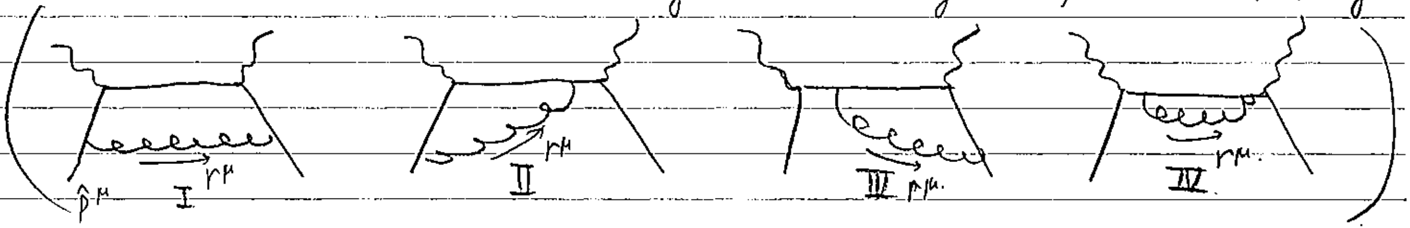
OPE makes it clear in the case of DIS.

Parton model + QCD 1-loop calculation.

$$\Rightarrow \quad F_1(x;Q^2) \cong \frac{1}{2}(Q_q)^2\left[f_q(x) + \int^1 \frac{dz}{z}\,\frac{\alpha_s}{2\pi}\,C_2 \int^{(p\cdot q)} \frac{d(p\cdot r)}{(p\cdot r)}\, f_q(z) \left\{ \frac{1+(x/z)^2}{1-(x/z)} \right\} \right]$$

+ anti-quark parton contributions
+ gluon parton contributions.

Logarithmic divergence from small $(p\cdot r)$ region.



If $f_q(z)$ is finite valued, then the observable $F_1(x;Q^2)$ is not.

Just like we have done in renormalization

$$\frac{4\pi}{g^2(p^2)} = \frac{4\pi}{g_0^2} + \frac{b}{2\pi}\ln\left(\frac{p}{\Lambda}\right) \Rightarrow \underbrace{\left[\frac{4\pi}{g_0^2} + \frac{b}{2\pi}\ln\left(\frac{\mu_R}{\Lambda}\right)\right]}_{=:\,(4\pi/g^2(\mu_R))} + \frac{b}{2\pi}\ln\left(\frac{p}{\mu_R}\right)$$

$\uparrow$ observable parameter $\qquad\qquad$ $\uparrow$ renormalized coupling finite.

$$\boxed{\underbrace{\left[f_q(x) + \int\frac{dz}{z}\int_0^{p\cdot q}\frac{d(p\cdot r)}{p\cdot r}\right]}_{\text{observable}} = \underbrace{f_q(x;\mu_F)}_{\text{"renormalized"}} + \int^1\frac{dz}{z}\,\frac{\alpha_s}{2\pi}\,C_2\int_{\mu_F^2}^{p\cdot q}\frac{d(p\cdot r)}{(p\cdot r)}\, f_q(z;\mu_F)\left\{\frac{1+(x/z)^2}{1-(x/z)}\right\}}$$

$-2\hat{p}\cdot r \sim (\hat{p}-r)^2$: virtuality of the parton q.

radiative corrections with virtuality $\lesssim \mu_F^2$ have been
(quantum effects)
(near light cone) swept under the carpet $(f(z;\mu_F^2))$

§12. Two Scale Problems and TMD Factorization.

§12.1. Two scale problems.

★ $\sigma(e^+e^- \to \text{hadron})_{tot}$ and $\left(\frac{d\sigma}{dx\, dQ^2}\right)_{DIS}$ at moderate x
(~~not $x \ll 1$~~)

are examples of single scale problems.

- set $s \gg \Lambda_{QCD}^2$ or $Q^2 \gg \Lambda_{QCD}^2$
- use $\mu_R \sim \sqrt{s}$ or $\sim Q$

$\left\{ \begin{array}{l} \text{perturbative calculation of OPE coefficients possible} \\ \alpha_s \ln(\sqrt{s}/\mu_R) \text{ is not large.} \end{array} \right.$

★ Many other observables in scattering experiments are functions of two different energy scales (or more).

Even when all the relevant energy scales are $\gg \Lambda_{QCD}$,
we may not be able to avoid $+\alpha_s \ln(E_1/E_2)$ appearing
in theoretical calculations (no matter how we choose μ_R).

- DIS at small x $\left(Q^2 \text{ and } (p\cdot q) \quad \text{ratio is } x\right)$
- two jet production cross section. $\left(\begin{array}{l} s \text{ and } p_{T.\text{cut}} \text{ for jet def.} \\ \text{ratio } \theta \end{array}\right)$
- Drell-Yan transverse mom. dependence

$$\left(\frac{d\sigma}{dy\, dQ^2\, d^2\vec{q}_T}\right)(h_1 + h_2 \to V^{(*)} + \text{anything})$$

vector boson $V^{(*)} = \gamma^*, Z^{(*)}, W^{(*)}$ (decay subsequently to leptons)

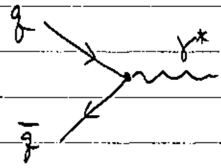
$q^\mu \Rightarrow \cdot\, Q^2 := q^2$
$= (q^0, q^3, \vec{q}_T) \quad \cdot\, y := \frac{1}{2}\ln\left(\frac{q^0+q^3}{q^0-q^3}\right)$

No.
Date

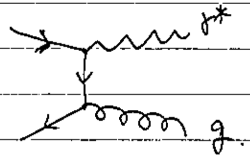
- When are such observables IR safe?
- In which sense do we say that they are safe?
- Even when they are safe, how do we resum perturbative series?

$\hookrightarrow$ Intimately related questions.

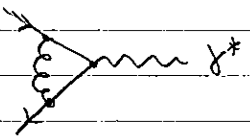
<u>Example</u> <u>Drell-Yan q_T-dependence.</u>

parton model. tree: 

$\Rightarrow \propto \delta^2(\vec{q}_T)$

at NLO. 

$\Rightarrow \propto \frac{\alpha_s}{\pi} C_2 \frac{1}{|\vec{q}_T|^2}$



$\Rightarrow \propto \delta^2(\vec{q}_T) \times \left(-\frac{\alpha_s}{\pi} C_2 \times \text{log.div.}\right)$

IR divergence expected to cancel after binning in $\vec{q}_T$

- need to resum $\alpha_s^n \ln^{2n}(Q^2/q_T^2)$
- partons in a hadron have intrinsic momenta. "$f_q(x, \vec{k}_T; \mu_F^2)$"?

$\longrightarrow$ not that simple actually.

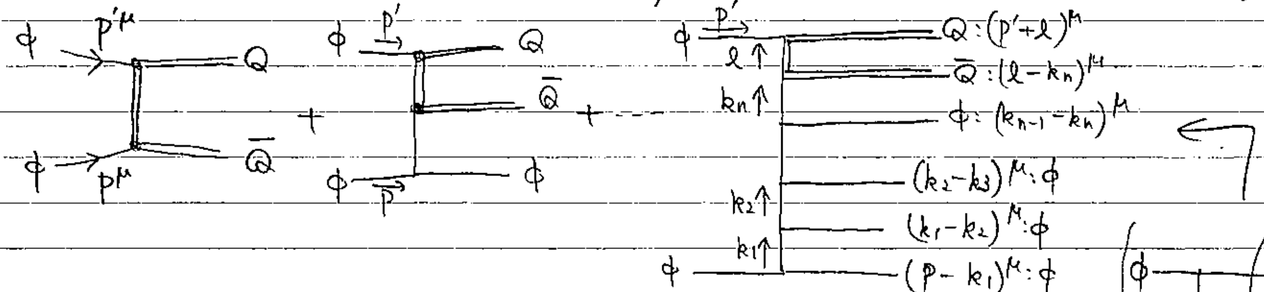
PLUS

§12.2 Double log phase space in 6D ϕ^3 theory

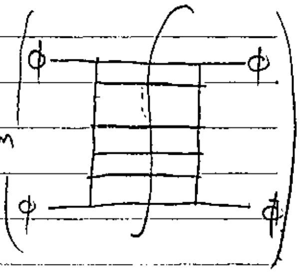
$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{g}{3!}\phi^3 \quad \Leftarrow \text{ IR dynamics similar to QCD}$$

$$-\lambda \phi Q\bar{Q} - M^2|Q|^2 \qquad [\phi] = +2 \qquad [g] = 0$$

At the parton level, think of $\mathcal{M}(\phi^{(*)} + \phi \to Q\bar{Q} + \text{anything})$



ladder diagram. Im



often dominate at higher order

$$2\,\mathrm{Im}(\overset{\text{forward}}{\mathcal{M}}) = \int \frac{d^6 l}{(2\pi)^6}\int\frac{d^6 k_n}{(2\pi)^6}\cdots\int\frac{d^6 k_1}{(2\pi)^6}\,(2\pi)\delta^+[(p'+l)^2 - M^2]\,(2\pi)\delta^+[(k_n - l)^2 - M^2]\,(2\pi)\delta^+[(k_{n-1}-k_n)^2]$$

$$\left|\frac{\lambda^2 g^n}{[l^2 - M^2][k_n^2][k_{n-1}^2]\cdots[k_2^2][k_1^2]}\right|^2 \quad (2\pi)\delta^+[(p-k_1)^2] \quad (2\pi)\delta^+[(k_1-k_2)^2]$$

Sudakov parameter

$$\begin{cases} -p'^\mu = 0\cdot p^\mu + (-1)\bar{p}^\mu \\ l^\mu = \rho_{n+1}\, p^\mu + \lambda_{n+1}\bar{p}^\mu + \vec{l}_T \\ k_n^\mu = \rho_n\, p^\mu + \lambda_n \bar{p}^\mu + \vec{p}_{T.n} \\ \vdots \\ k_2^\mu = \rho_2\, p^\mu + \lambda_2\bar{p}^\mu + \vec{p}_{T.2} \\ k_1^\mu = \rho_1\, p^\mu + \lambda_1 \bar{p}^\mu + \vec{p}_{T.1} \\ p^\mu = 1\cdot p^\mu + 0\cdot\bar{p}^\mu \end{cases}$$

- $\delta^+[(k_{i-1}-k_i)^2] = \delta^+[(\rho_{i-1}-\rho_i)(\lambda_{i-1}-\lambda_i)s - |\vec{k}_i - \vec{k}_{i-1}|^2]$
  $\Rightarrow (\rho_{i-1} > \rho_i)\ \&\ (\lambda_{i-1} > \lambda_i)\ \&\ \delta[\text{fcn}]$
- $\begin{cases} \rho_{n+1} > 0, & \lambda_{n+1} > -1, \\ \rho_n > \rho_{n+1}, & \lambda_n > \lambda_{n+1}, \\ 1 > \rho_1, & 0 > \lambda_1 \end{cases}$ $\quad$ (if $\bar{p}^\mu = p'^\mu$ in DIS)

$$\Rightarrow \boxed{\begin{array}{l} 1 > \rho_1 > \rho_2 > \cdots > \rho_n > \rho_{n+1} > (x > 0) \\ 0 > \lambda_1 > \lambda_2 > \cdots > \lambda_n > \lambda_{n+1} > -1 \end{array}}$$

$\bar{p}^\mu$: light like

$2p\cdot\bar{p} = s$

assume hierarchical $\Rightarrow \delta[(-\lambda_i)\rho_{i-1}s - |\vec{k}_i - \vec{k}_{i-1}|^2]$

$$= \frac{1}{s\rho_{i-1}}\,\delta\left[-\lambda_i - \frac{|\vec{k}_i - \vec{k}_{i-1}|^2}{s\,\rho_{i-1}}\right]$$

So, the ladder diagram contributes by

$$\Delta(2\,\mathrm{Im}\,\mathcal{M}) = \frac{\lambda^4}{4\pi}\,\alpha_g^{\,n}\int\frac{d\rho_{n+1}}{\rho_{n+1}}\int\frac{d\rho_n}{\rho_n}\cdots\int\frac{d\rho_1}{\rho_1}(2\pi)\delta\left(s\rho_{n+1}-|\vec{l}_T|^2-M^2\right)\int\frac{d^2 l_T}{(2\pi)^4}\sim\int\frac{d^2 \vec{k}_{1,T}}{(2\pi)^4}\cdots\frac{1}{|\vec{l}_T|^4\,|\vec{k}_{T,n}|^4\cdots|\vec{k}_{T,2}|^4\,|\vec{k}_{T,1}|^4}$$

$$\left(1\gg\rho_1\gg\rho_2\gg\cdots\gg\rho_{n+1}\gg\frac{M_a^2}{s}\right)$$

$\left(k_i^2 = s\rho_i(\lambda_i)-|\vec{k}_{i,T}|^2\right.$; $s\rho_i|\lambda_i| \ll s\rho_i|\lambda_{i+1}| \sim |\vec{k}_{T,i}-\vec{k}_{T,i+1}|^2$

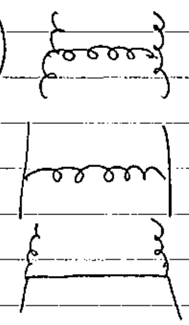
~~$\ll s\rho_{i-1}|\lambda_{i-1}| \sim |\vec{k}_{T,i}-\vec{k}_{T,i-1}|^2$~~.

$\rightarrow$ ignore $s\rho_i|\lambda_i|$ against $|\vec{k}_{T,i}|^2$.

$$\Rightarrow \frac{\lambda^4}{2}\,\frac{\ln^{n+1}(s/M_a^2)}{(n+1)!}\,\alpha_g^{\,n}\,\frac{\ln^{n+1}(Q^2/k_{T,cut}^2)}{(4\pi)^2}$$

double logarithms

QCD
- gluon ladder : $d^2\vec{k}_T\,\frac{1}{|\vec{k}_T^2|}\,\frac{1}{|\vec{k}_T^2|}\times\left((\partial\kappa)^2\leftarrow\text{vertex}\right)$
- fermion + glue ladder : $d^2\vec{k}_T\,\frac{1}{k_T}\,\frac{1}{k_T}$
- $d^2k_T\,\frac{1}{k_T^2}\,\frac{1}{k_T^2}\,\not{k}$.



What is $k_{cut,T}$? (necessary for a sensible results)

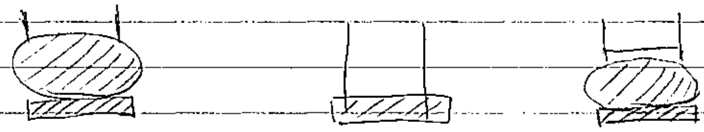
- Make it a rule not to include $|\vec{k}_T|<\mu_I$ DOF in the loop
- Replace the initial single parton state $\phi(p^\mu)$ by a distribution $f(p,\vec{k}_T;\mu_I)$

Bethe-Salpeter equation / Balitsky Fadin Kuraev Lipatov eq.

Let $F(\rho, \lambda, \vec{k}_T ; \mu_I) \overset{?}{=} \int \langle h(\vec{p})| \, \phi(-x/2)\, \phi(x/2) \, |h(\vec{p})\rangle \, e^{+i(p^- x \rho + \vec{p}^- x \lambda - \vec{k}_T \cdot \vec{x}_T)} \, d^6 x .$

Then we say that

$$\boxed{F(\rho, \lambda, \vec{k}_T ; \mu_I) \simeq F(\rho, \lambda, \vec{k}_T ; \mu_0) + \frac{g^2}{\vec{k}_T^{\,2} \vec{k}_T^{\,2}} \frac{S}{2} \int \frac{d\rho'}{2\pi} \int \frac{d\lambda'}{2\pi} \int^{\mu_I} \frac{d^4 \vec{k}_T'}{(2\pi)^4} (2\pi) \delta^+ [S \rho' \lambda - |\vec{k}_T - \vec{k}_T'|^2] \, F(\rho', \lambda', \vec{k}_T' ; \mu_I)}$$



$d\lambda$.

$$\boxed{\mathcal{F}(\rho, \vec{k}_T ; \mu_I) \simeq \mathcal{F}(\rho, \vec{k}_T ; \mu_0) + \frac{\alpha_g}{k_T^2 k_T^2} \int_\rho^1 \frac{d\rho'}{\rho'} \int_{\mu_0}^{\mu} \frac{d^4 \vec{k}_T'}{(2\pi)^4} \mathcal{F}(\rho', \vec{k}_T' ; \mu_I)}$$

Assume that $\mathcal{F}(\rho, \vec{k}_T ; \mu_I) \sim \frac{f(\rho ; \mu_I)}{|k_T|^4}$ $\quad \mathcal{F}(\rho, \vec{k}_T ; \mu_0) = \frac{f(\rho ; \mu_0)}{|k_T|^4}$

Then $\int_0^1 d\rho \, \rho^{j-1}$ (Mellin transformation)

$$\tilde{f}(j ; \mu_I) \simeq \tilde{f}(j ; \mu_0) + \frac{\alpha_g \ln(\mu_I^2/\mu_0^2)}{(4\pi)^2} \frac{1}{j} \tilde{f}(j ; \mu_I)$$

$$\Rightarrow \boxed{\tilde{f}(j ; \mu_I) \simeq \frac{j \, \tilde{f}(j ; \mu_0)}{j - \frac{\alpha_g \ln(\mu_I^2/\mu_0^2)}{(4\pi)^2}}}$$

Take the inverse Mellin transformation

$$\boxed{\mathcal{F}(\rho, \vec{k}_T ; \mu_I) \sim \rho^{-\left[\frac{\alpha_g \ln(\mu_I^2/\mu_0^2)}{(4\pi)^2}\right]} \Big/ (k_T^2 k_T^2) \times \tilde{f}\left(\frac{\alpha_g}{(4\pi)^2} \ln\left(\frac{\mu_I^2}{\mu_0^2}\right), \mu_0\right)}$$

by picking up the pole $j = \frac{\alpha_g}{(4\pi)^2} \ln(\mu_I^2/\mu_0^2)$