# §12.3 Factorization for the Drell-Yan Diff. Cross Section

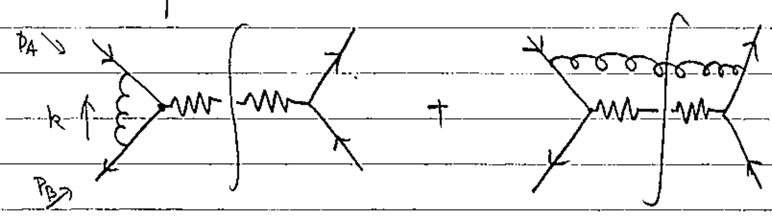
Consider (having Drell-Yan process in mind)

$$(*) := \int d\Pi_X \, \langle q(\vec{p}_A)\,\bar{q}(\vec{p}_B)\,|\,J^\mu(0)\,|\,X\rangle\langle X\,|\int d^4x\, J^\nu(x)\, e^{iq\cdot x}\,|\,q(\vec{p}_A)\,\bar{q}(\vec{p}_B)\rangle .$$

At tree level

$$(*) = \left[\bar{u}(\vec{p}_A)\gamma^\mu v(\vec{p}_B)\right]\left[\bar{v}(\vec{p}_B)\gamma^\nu u(\vec{p}_A)\right](2\pi)^4\delta^3(\vec{p}_A+\vec{p}_B-\vec{q})\,\delta(E_A+E_B-q^0).$$

At 1-loop level:

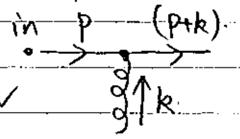



soft modes give rise to double log corrections.

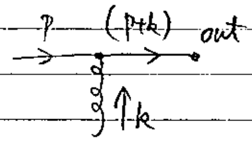
How do they cancel?

$\boxed{\text{Grammer - Yennie approximation}}$

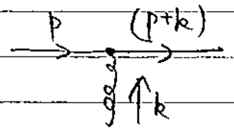
☆ soft or collinear gluon attached to a $q$-line




$$i[\not{p}+\not{k}]\,(-ig\gamma^\kappa)\,u(\vec{p}) = g\Big\{\underbrace{2(p+k)^\kappa}_{\text{parallel to } p^\mu} u(\vec{p}) - \gamma^\kappa \underbrace{(\not{p}+\not{k})\,u(\vec{p})}_{\approx 0 \text{ Dirac eq.}}\Big\}$$



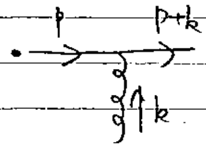

$$\bar{u}(\vec{p}+\vec{k})\,(-ig\gamma^\kappa)\,i[\not{p}] = g\Big\{\bar{u}(\vec{p}+\vec{k})\,2p^\kappa - \underbrace{\bar{u}(\vec{p}+\vec{k})\,\not{p}}_{\approx \text{ Dirac eq.}}\gamma^\kappa\Big\}$$



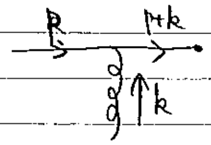

$$i[\not{p}+\not{k}]\,(-ig\gamma^\kappa)\,i[\not{p}] = g\Big\{2p^\kappa\, i[\not{p}+\not{k}] - \underbrace{i(\not{p}+\not{k})\,\not{p}}_{\approx p^2 \text{ or } (p+k)\cdot p \approx 0}\gamma^\kappa\Big\}$$

light like.

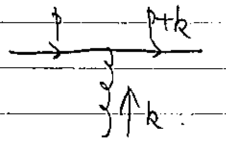
## ☆ soft or collinear gluon attached to an $\bar{q}$-line.




$$\bar{v}(\vec{p})(-ig\gamma^{\kappa})\, i[-(\cancel{p+k})] = -g\{\bar{v}(\vec{p})\, 2(p+k)^{\kappa} - \underbrace{\bar{v}(\vec{p})(\cancel{p+k})}_{\approx 0 \text{ Dirac eq.}}\gamma^{\kappa}\}$$




$$i[-\cancel{p}](-ig\gamma^{\kappa})\, v(\overrightarrow{p+k}) = -g\{2p^{\kappa} v(\overrightarrow{p+k}) - \gamma^{\kappa} \underbrace{\cancel{p}\, v(\overrightarrow{p+k})}_{\approx 0 \text{ Dirac eq.}}\}$$




$$i[-\cancel{p}](-ig\gamma^{\kappa})\, i[-(\cancel{p+k})] = -g\{\underbrace{2p^{\kappa}[i(\cancel{p+k})]}_{\substack{\text{dominant contrib.} \\ \propto\, p^{\mu}}} - \gamma^{\kappa}(-i)\underbrace{\cancel{p}(\cancel{p+k})}_{\approx 0 \ (\text{light like})}\}$$

## ☆ soft or collinear gluon attached to a gluon line.

~~skipped in this lecture (note)~~

## ☆ Denominators of the propators.

$$[(p+k)^2 - M^2 + i\varepsilon] = \overbrace{(p^2 - M^2)}^{0} + 2p\cdot k + k^2$$

$$\begin{cases} \text{soft } k^{\mu} & \Rightarrow \quad \underset{\underline{\underline{O(\lambda)}}}{+2p^{+}k^{-}} + \underset{O(\lambda^2)}{2k^{+}k^{-}} - \underset{O(\lambda^2)}{\vec{k}_T^{\,2}} \quad \Rightarrow [2p\cdot k + i\varepsilon] \\[2ex] \text{collinear to } p^{\mu} & \Rightarrow \quad \underset{\underline{O(\lambda^2)}}{2p^{+}k^{-}} + \underset{\underline{O(\lambda^2)}}{2k^{+}k^{-}} - \underset{\underline{O(\lambda^2)}}{\vec{k}_T^{\,2}} \quad \Rightarrow [2p\cdot k + k^2 + i\varepsilon] \\[2ex] \text{anti-collinear to } p^{\mu} & \Rightarrow \quad \underline{\underline{O(1)}} \quad O(\lambda^2) \quad O(\lambda^2) \quad \Rightarrow [2p\cdot k + i\varepsilon] \end{cases}$$

## §12.3.1 Soft contributions. (using the Grammer-Yennie approx.)

gluon momentum $k^\mu$ flowing into a $q$ / $\bar{q}$ line

Soft gluon coupled to

◎ an incoming $q$ line before the cut : $(-igt^a)\dfrac{i2p^\kappa}{2p\cdot k+i\varepsilon}\times\left[\dfrac{-i\eta_{\kappa\lambda}}{k^2+i\varepsilon}\right]e^{-ik\cdot x}$

◦ an outgoing $q$ line after the cut : $(igt^a)\dfrac{-i2p^\kappa}{-2p\cdot k-i\varepsilon}\times\left[\dfrac{-i\eta_{\kappa\lambda}}{k^2+i\varepsilon}\right]e^{-ik\cdot x}$

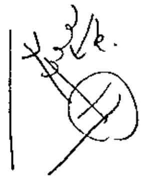
$\Longleftrightarrow$ insert an eff. operator
$$\begin{cases} -igt^a\displaystyle\int_{-\infty}^{0} n^\kappa\varepsilon^a_\kappa\, e^{-i(k\cdot n+i\varepsilon)a}\,da\, e^{-ik\cdot x} & \Leftarrow \boxed{W[x,x-\infty(n)] := P\exp\left[-ig\int_{-\infty}^{0}da\, n^\kappa A_\kappa(x+na)\right]} \\ igt^a\displaystyle\int_{-\infty}^{0} n^\kappa\varepsilon^a_\kappa\, e^{-i(k\cdot n+i\varepsilon)}\, e^{-ik\cdot x}\,da & \Leftarrow \boxed{\bar{W}[x,x-\infty(n)] := \bar{P}\exp\left[+ig\int_{-\infty}^{0}da\, n^\kappa A_\kappa(x+na)\right]} \end{cases}$$

note
$$\bar{W}[x_f,x_i] = \left(W[x_f,x_i]\right)^\dagger = W[x_i,x_f]$$
unitary

◦ an incoming $\bar{q}$-line before the cut $(igt^a)\dfrac{ip^\kappa}{p\cdot k+i\varepsilon}\times\left[\dfrac{-i\eta_{\kappa\lambda}}{k^2+i\varepsilon}\right]e^{-ik\cdot x}$

◦ an outgoing $\bar{q}$-line after the cut $(-igt^a)\dfrac{-ip^\kappa}{-p\cdot k-i\varepsilon}\times\left[\dfrac{-i\eta_{\kappa\lambda}}{k^2+i\varepsilon}\right]e^{-ik\cdot x}$.

$\Longleftrightarrow$ insert an eff. operators
$$\begin{cases} (igt^a)\displaystyle\int_{-\infty}^{0} e^{-i(k\cdot n+i\varepsilon)a}\,\varepsilon^a_\kappa n^\kappa\,da & \Leftarrow \boxed{\bar{W}[0,-\infty(n)] = W[\infty(-n),0]} \\ (-igt^a)\displaystyle\int_{-\infty}^{0} e^{-i(k\cdot n+i\varepsilon)a}\,\varepsilon^a_\kappa n^\kappa\,da & \Leftarrow \boxed{W[0,-\infty(n)]} \end{cases}$$



To an outgoing $q$-line before the cut.

$$\cdot\ (-igt^a)\frac{ip^\kappa}{-p\cdot k+i\varepsilon}\times\left[\frac{-i\eta_{\kappa\lambda}}{k^2+i\varepsilon}\right]$$

$$\Longleftrightarrow\ -igt^a\int_0^{+\infty}e^{-i(k\cdot n-i\varepsilon)a}\,\varepsilon^a_\kappa n^\kappa\,da\ \Leftarrow\ W[\infty(n),0]$$

incoming $q$-

$$\cdot\ (igt^a)\frac{-ip^\kappa}{p\cdot k-i\varepsilon}\times\left[\frac{-i\eta_{\kappa\lambda}}{(k^2+i\varepsilon)}\right]$$

$$\Longleftrightarrow\ igt^a\int_0^{+\infty}e^{-i(k\cdot n-i\varepsilon)a}\,\varepsilon_\kappa n^\kappa\,da\ \Leftarrow\ \overline{W}[\infty(n),0]$$

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To an outgoing $\bar{q}$ line before the cut: $(igt^a)\dfrac{ip^\kappa}{-p\cdot k+i\varepsilon}\times\left[\dfrac{-i\eta_{\kappa\lambda}}{k^2+i\varepsilon}\right].$

insert

$$\Longleftrightarrow\ +igt^a\int_0^{+\infty}e^{-i(k\cdot n-i\varepsilon)a}\,\varepsilon^a_\mu n^\mu\,da\ \Leftarrow\ \overline{W}[\infty(n),0].$$

To an incoming $\bar{q}$-line after the cut: $(-igt^a)\dfrac{-ip^\kappa}{p\cdot k-i\varepsilon}\times\left[\dfrac{-i\eta_{\kappa\lambda}}{k^2+i\varepsilon}\right]$

insert

$$\Longleftrightarrow\ -igt^a\int_0^{+\infty}e^{-i(k\cdot n-i\varepsilon)a}\,da\,\varepsilon^a_\kappa n^\kappa\ \Leftarrow\ W[\infty(n),0].$$

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## Summary / Intuition

$i\mathcal{L}_{int} \supset -ig A^b_\mu (\bar{\Psi}^b \gamma^\mu \Psi^b)$

Before the cut — sourced by $\Downarrow$ the world line

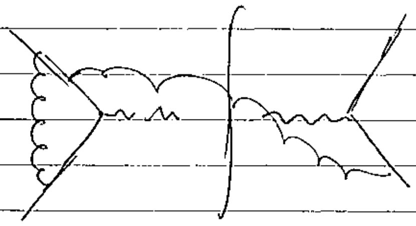
$$\left|\begin{array}{l}
\text{incoming } q : W[x_b, x_b - \infty(n)] = P\exp\left[-ig t^b \int_{-\infty}^{0} da\, n^\kappa A^b_\kappa (x_b + an)\right] \\
\text{incoming } \bar{q} : \bar{W}[x_b, x_b - \infty(n)] = \bar{P}\exp\left[ig t^b \int_{-\infty}^{0} da\, n^\kappa A^b_\kappa (x_b + an)\right] \\
\text{outgoing } q : W[\infty(n) + x_b, x_b] = P\exp\left[-ig \int_{x_b}^{x_b+\infty(n)} dx^\kappa A_\kappa\right] \\
\text{outgoing } \bar{q} : \bar{W}[\infty(n) + x_b, x_b] = \bar{P}\exp\left[ig \int_{x_b}^{x_b+\infty(n)} dx^\kappa A_\kappa\right]
\end{array}\right|$$

After the cut $\Longrightarrow$ take Hermitian conjugate

Include the quantum corrections due to the soft corrections.

$$(*) = \int d^4x \int d\Pi_X^{\text{non-soft}} \langle h(\vec{p}_A) h'(\vec{p}_B) | J^\mu(0) | X' \rangle \langle X' | J^\nu(x) | h(\vec{p}_A) h'(\vec{p}_B) \rangle e^{iq\cdot x}$$

$$\times \langle \Omega | \bar{W}[0, -\infty(n_A)] W[0, -\infty(n_B)] \bar{W}[x, x-\infty(n_B)] W[x, x-\infty(n_A)] | \Omega \rangle \quad \boxed{\text{soft only}} \; (E_0)$$



- $x^\pm$-dep : from the first line.
- $\vec{x}_T$-dep : $\begin{cases} \text{from the both} & \text{if } q_T \sim k_T \sim E_{cut} \\ \text{from the first line} & \text{if } q_T \gg k_T \sim E_{cut}. \end{cases}$

$$S_0(\vec{x}_T) := \langle \Omega | \mathrm{Tr}_{N_c} \left[ \bar{W}[0, -\infty(n_A)] W[0, -\infty(n_B)] \bar{W}[(0^\pm, \vec{x}_T), (0^\pm, \vec{x}_T) - \infty(n_B)] W[(0^\pm, \vec{x}_T), (0^\pm, \vec{x}_T) - \infty(n_A)] \right] | \Omega \rangle .$$

$$(*) = \int d^2\vec{x}_T \left[ \int_{\text{non-soft}} d\Pi_X' \int d^2x^\pm \langle hh' | J^\mu(0) | X' \rangle \langle X' | J^\nu(x) | hh' \rangle e^{iq^\pm \cdot x^\mp - i\vec{q}_T \cdot \vec{x}_T} \right] \boxed{S_0(\vec{x}_T)}.$$

$\boxed{\text{factorization.}}$ (non-soft) $\times$ (soft).

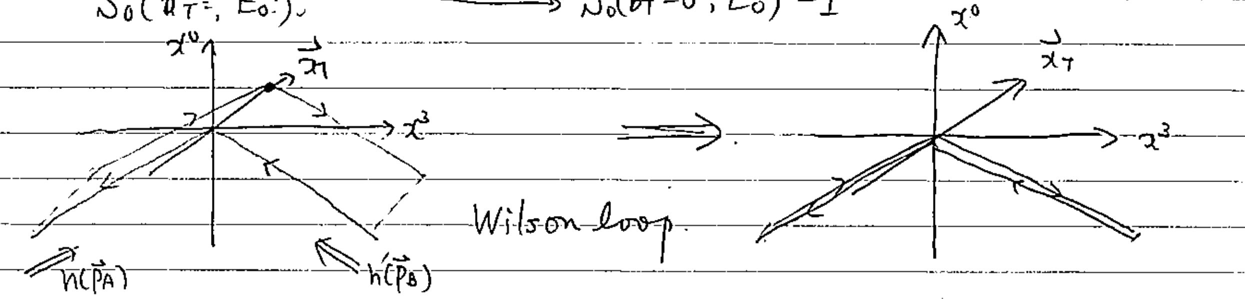
## $\boxed{\text{Observations}}$

◎ $$\int \frac{d^2\vec{q}_T}{(2\pi)^2}\,(*) = \int \frac{d^2\vec{q}_T}{(2\pi)^2}\int d^2\vec{x}_T\, e^{i\vec{q}_T\cdot\vec{x}_T}\, S_0(\vec{x}_T; E_0)$$
$$\times\int d^2x^\pm \int d\Pi_{X'}(E_0)\, \langle h h'| J^\mu(0)|X'\rangle\langle X'|J^\nu(x)|hh'\rangle\, e^{iq^\pm\cdot x^\mp}$$

cross section
$q_T$-diff ⟶ $q_T$-integrated

$$= \int d^2x^\pm \int d\Pi_{X'}(E_0)\,\langle hh'|J^\mu(0)|X'\rangle\langle X'|J^\nu(x^\pm, \vec{x}_T=\vec 0)|hh'\rangle\, e^{iq^\pm\cdot x^\mp}\, S_0(\vec{x}_T=\vec 0; E_0)$$

$S_0(\vec{x}_T; E_0)$ ⟶ $S_0(\vec b_T=\vec 0; E_0)=1$ ← unitarity




◎ Even in $q_T$-diff cross section,

the soft factor $S_0(\vec b_T;E_0)$ is free from soft divergence.

$$\langle\Omega|T\{\overline{W}[0,-\infty(n_B)]\,W[0,-\infty(n_A)]\}|\Omega\rangle = 1 - \frac{\alpha_g}{2\pi}C_2\ln\left(\frac{E_0^2}{m_g^2}\right)(\Delta y)$$

virtual corr. (Sudakov factor)

$$\langle\Omega|T\{W[-\vec b_T, -\vec b_T-\infty(n_B)]\,W[0,-\infty(n_A)]\}|\Omega\rangle = 1+\frac{\alpha_g}{2\pi}C_2\left(\int_{m_g^2(k_T^2)}^{E_0^2} d(k_T^2)\, \underbrace{J_0(b_Tk_T)}_{\frac{1}{2\pi}\int_0^{2\pi} d\theta\, e^{i\vec b_T\cdot\vec k_T}}\right)(\Delta y)$$

real soft emission

$\ln\left(\frac{1}{m_g^2 b_T^2}\right)$

- $S_0(\vec b_T, E_0) \sim 1 - \frac{\alpha_g}{\pi}C_2\ln\left(E_0^2 b_T^2\right)\times(\Delta y)$
- $d^2\vec b_T$ integrated : dominated by $b_T\sim 1/q_T$.

Long distance (small $k_T$) modes cannot distinguish the two parallel Wilson lines $W[-\vec b_T, -\vec b_T-\infty(n_B)]\,\overline{W}[0,0-\infty(n_B)]$.

$$(\Delta y) := \frac12\left\{\ln\left(\frac{n_A^+}{n_A^-}\right)-\ln\left(\frac{n_B^+}{n_B^-}\right)\right\}$$ diverges if we choose $n_A^\mu, n_B^\mu$ light-like.

⟹ choose them to be slightly spacelike. (for DY)

$S_0(\vec b_T; E_0, v_1^\mu, v_2^\mu)$ $\qquad \Delta y = \frac12\ln\left[\frac{(v_1\cdot v_2)^2}{(v_1)^2(v_2)^2}\right]$