Drell-Yan process at the LO in QCD in an OPE-like language

$$d\sigma_{DY} = d^4q \begin{bmatrix} \text{a factor} \\ \text{mass-dim} = -2 \end{bmatrix}_{\mu\nu} \times \frac{1}{2S_{AB}} \times (*)^{\mu\nu}$$

$$(*)^{\mu\nu} = \int d\Pi_X \int \langle h_A(\vec{p}_A) h_B(\vec{p}_B) | J^\mu(0) | X \rangle \langle X | J^\nu(x) | h_A(\vec{p}_A) h_B(\vec{p}_B) \rangle e^{iq\cdot x} d^4x$$

$$\Rightarrow \int d\Pi_X \int \langle h_A . h_B | J^\mu(0) | X \rangle \langle X | \bar{\psi}(x^+, \cancel{x^-}, \vec{b}_T) \gamma^\nu \psi(\cancel{x^+}, x^-, \vec{b}_T) | h_A h_B \rangle e^{i(q^+x^- + q^-x^+ - \vec{q}\cdot\vec{b})} d^4x .$$

$$\left(\begin{cases} 2q^+q^- =: Q^2 \\ q^+/q^- =: e^{2y} \end{cases} \Rightarrow^{\text{at LO}} (q^+, q^-) = \begin{matrix} (\sqrt{2}E x_A, 0) \\ + \\ (0, \sqrt{2}E x_B) \end{matrix} \iff \begin{matrix} x_A x_B = Q^2/S \\ x_A/x_B = e^{2y} \end{matrix} \right.$$

$$= \sqrt{\tfrac{S}{2}}(x_A, x_B) \qquad \left. \begin{cases} x_A = \frac{Q}{\sqrt{S}} e^{y} \\ x_B = \frac{Q}{\sqrt{S}} e^{-y} \end{cases} \right)$$

$$\sim (\gamma^\mu)_{ac} (\gamma^\nu)_{db} \int d^2\vec{b}_T\, e^{-i\vec{q}_T\cdot\vec{b}_T} \int dx^- \langle h_A(\vec{p}_A) | \bar{\psi}_a(0)\, \psi_b(x^-, \vec{b}_T) | h_A(\vec{p}_A) \rangle e^{i\sqrt{\frac{S}{2}} x^- x_A}$$
$$\int dx^+ \langle h_B(\vec{p}_B) | \psi_c(0)\, \bar{\psi}_d(x^+, \vec{b}_T) | h_B(\vec{p}_B) \rangle e^{i\sqrt{\frac{S}{2}} x^+ x_B}$$

$$\sqrt{S/2}\, x^- =: \kappa_A . \quad \sqrt{S/2}\, x^+ =: \kappa_B$$

$$= (\gamma^\mu)_{ac} (\gamma^\nu)_{db} \int d^2\vec{b}_T\, e^{-i\vec{q}_T\cdot\vec{b}_T} \frac{2}{S} \int d\kappa_A \langle h_A | \bar{\psi}_a(0)\, \psi_b(x^- = \sqrt{\tfrac{2}{S}}\kappa_A, \vec{b}_T) | h_A \rangle e^{i\kappa_A x_A}$$
$$\int d\kappa_B \langle h_B | \psi_c(0)\, \bar{\psi}_d(x^+ = \sqrt{\tfrac{2}{S}}\kappa_B, \vec{b}_T) | h_B \rangle e^{i\kappa_B x_B} .$$

$$\Leftarrow \pi^2 \frac{2}{S} \int d^2\vec{b}_T\, e^{-i\vec{q}_T\cdot\vec{b}_T}\, \mathrm{Tr}_{4\times4}\left[\gamma^\nu \frac{\not{p}_A}{2} \gamma^\mu \frac{\not{p}_B}{2}\right] \int \frac{d\kappa_A}{4\pi} \langle h_A | \bar{\psi}(0) \not{\bar{n}}_A \psi(x^- = \sqrt{\tfrac{2}{S}}\kappa_A, \vec{b}_T) | h_A \rangle e^{i\kappa_A x_A}$$
$$\underbrace{\int \frac{d\kappa_B}{4\pi} \langle h_B | \psi(0) \not{\bar{n}}_B \psi(x^+ = \sqrt{\tfrac{2}{S}}\kappa_B, \vec{b}_T) | h_B \rangle e^{i\kappa_B x_B}}$$

$\left[\begin{array}{l} \bar{n}_A^\mu : \text{light-like vector} \\ \text{satisfying } p_A\cdot\bar{n}_A = +1 \\ (\text{also } p_B\cdot\bar{n}_B = +1) \end{array} \right]$ $\approx$ PDF $f_{q/h_A}(x_A)\ f_{\bar{q}/h_B}(x_B)$

now ... with $\vec{b}_T$-dependence

~~If interested in the $d^2\vec{q}_T$-integrated cross section~~

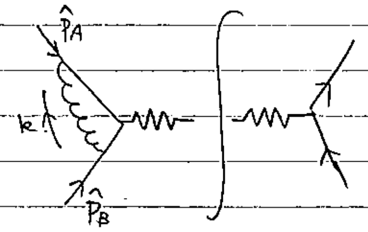
~~$\rightarrow d^2\vec{b}_T\ \delta^2(\vec{b}_T) \times (\text{any } \vec{b}_T\text{-dependence})$ just set $\vec{b}_T = \vec{0}$.~~

Factorization of soft contributions

$$\boxed{(*)^{\mu\nu} = \int d^2x^{\pm}\, d^2\vec{b}_T\; e^{i q^{\mp} x^{\mp}}\, e^{-i\vec{q}_T\cdot\vec{b}_T}\; S_0(\vec{b}_T)\times\int \overbrace{d\Pi_{X'}}^{\text{non-soft}} \langle h_A h_B|J^{\mu}(0)|X'\rangle\langle X'|J^{\nu}(x)|h_A h_B\rangle}$$

Next, focus on contributions from DOFs collinear to h_A

◦ Virtual correction due to a gluon w/ k^μ collinear to h_A.



$$\left[\bar{v}(\hat{p}_B)(-ig t^a\gamma^\kappa)\frac{i(\cancel{k}-\hat{\cancel{p}}_B)}{(k-\hat{p}_B)^2+i\epsilon}\gamma^\mu\frac{i(\hat{\cancel{p}}_A+\cancel{k})}{(\hat{p}_A+k)^2+i\epsilon}(-ig t^b\gamma^\lambda)u(\hat{p}_A)\right]\frac{-i\eta_{\kappa\lambda}\delta_{ab}}{k^2+i\epsilon}$$

$\downarrow$ with k^μ collinear to $\hat{p}_A^\mu \parallel p_A^\mu$

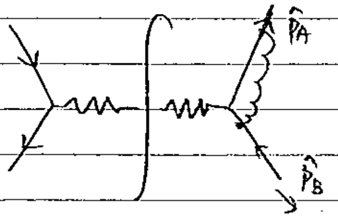
$[\lambda = -\text{ dominates}]$ while $\left[\begin{matrix}(k^+\gamma^-)\\ (\hat{p}_B^-\gamma^+)\end{matrix}\text{ comparable}\right] \Rightarrow \boxed{\text{just keep } (\gamma^{\kappa=+}) \text{ and } (\gamma^{\lambda=-})}$

$$\Rightarrow\; g^2 C_2\left[\bar{v}(\hat{p}_B)\gamma^\mu\frac{(\hat{\cancel{p}}_A+\cancel{k})}{(\hat{p}_A+k)^2+i\epsilon}\gamma^\lambda u(\hat{p}_A)\right]\frac{-i\eta_{\lambda\kappa}}{k^2+i\epsilon}\,\frac{n_B^\kappa}{n_B\cdot k-i\epsilon}$$

Grammer-Yennie approx.

∴ The incoming $\bar{q}$-line can be replaced by $\overline{W}[(x^-,\vec{b}_T),(x^-,\vec{b}_T)-\infty(n_B)]$.

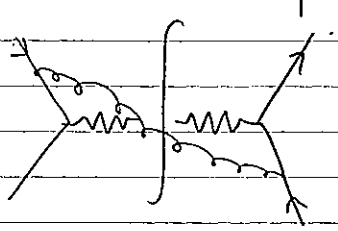
another
◦ Virtual correction due to a gluon w/ k^μ collinear to h_A



The outgoing $\bar{q}$-line can be replaced by

$W[0,-\infty(n_B)] = \overline{W}[\infty(-n_B),0]$.

• real emission of a gluon collinear to h_A



The outgoing $\bar{q}$-line can be replaced by

$W[0,-\infty(n_B)] = \overline{W}[\infty(-n_B),0]$ as above.

PLUS

The "collinear to h_A" contributions:

$$\underset{\text{coll.}A}{\Delta}\left(\int^{\text{non-soft}} d\Pi_X \; \langle h_A h_B | J^\mu(0) | X' \rangle \langle X' | J^\nu(x) | h_A h_B \rangle \right)$$

$$= \int^{\substack{\text{non-coll}A \\ \text{non-soft}}} d\Pi_{X''} \; \langle h_B | \left(\gamma^\mu \psi(0)\right)_a | X'' \rangle \langle X'' | \left(\bar\psi(x)\gamma^\nu\right)_b | h_B \rangle$$

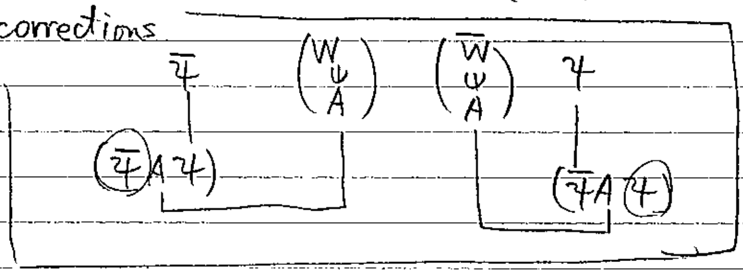
($a, b = 1\sim 4$, Dirac spinor component)

$$\times \left(\langle h_A(\vec p_A) | \bar\psi_a(0)\, W[0, -\infty(n_B)]\, \bar W[(x^-, \vec b_T), (x^-, \vec b_T) - \infty(n_B)]\, \psi_b(x^-, \vec b_T) | h_A(\vec p_A) \rangle \right)$$

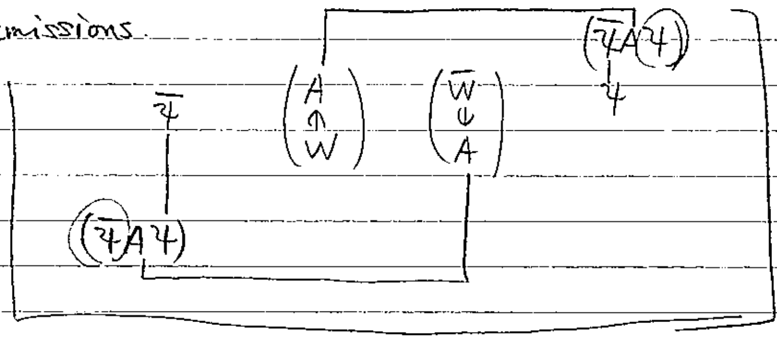
$$\qquad \| \qquad\qquad \text{path integral only w/ } k^\mu \text{ coll. to } h_A$$

$$J_{A,0}(x^-, \vec b_T)$$

Virtual corrections



Real emissions



factorization of the contributions collinear to h_A.

using an expression (factor) similar to $\boxed{\text{PDF.}}$

("need to say something about the flow of color, though ...)

The same argument applies to contributions with k^μ collinear to h_B.

$\boxed{\text{Fourier transform of } J_A(x^-, \vec b_T) \approx \text{transverse momentum-dep. PDF.}}$

$$\boxed{\begin{aligned}(*)^{\mu\nu} &= \int dx^+ dx^- d^2\vec{b}_T\, e^{iq\cdot x}\, S_0(\vec{b}_T)\, J_{A,0}(x^-,\vec{b}_T)\, J_{B,0}(x^+,\vec{b}_T)\, H_0(x^+,x^-,\vec{b}_T) \\ H(x^+,x^-,\vec{b}_T) &\sim \int_{\substack{\text{non-soft}\\ \text{non-col } A/B}} d\Pi_{X''} \langle h_A(\vec{p}_A) h_B(\vec{p}_B) | J^\mu(0) | X'' \rangle \langle X'' | J^\nu(x) | h_A h_B \rangle\end{aligned}}$$

The idea of the factorization formula for $\frac{d\sigma_{DY}}{d^4q}$.

This is not the end of the story because we need to think of...

$$\left\{\begin{array}{l}\cdot \text{ gauge-invariant def. of the factors } S_0,\ J_{A/B,0},\ H_0 \\ \quad \text{(without relying upon momentum cut-off)} \\ \cdot \text{ how to avoid double counting.}\end{array}\right\}$$

In $S_0 := \langle \Omega | T\left\{ \frac{1}{N_c} \mathrm{Tr}\left(\overline{W}[0,-\infty(n_A)]\, W[0,-\infty(n_B)]\, \overline{W}[\vec{b}_T, \vec{b}_T-\infty(n_B)]\, W[\vec{b}_T, \vec{b}_T-\infty(n_A)] \right)\right\} |\Omega\rangle$

(path integral only in the soft region.)

we can use dimensional regularization.

for d^2k_T: $\boxed{\frac{d^2\vec{k}_T}{(2\pi)^2} \Rightarrow \frac{d^n\vec{k}_T}{(2\pi)^n}\mu^{2-n}}$ $+\overline{MS}$ scheme.

$\Rightarrow$ only $k_T \leq \mu$ contribute.

The rapidity divergence (when n_A^μ, n_B^μ are light like)

$\Longleftrightarrow \langle\Omega| T\{\bar{W}[0, -\infty(n_B)]\, W[0, -\infty(n_A)]\}|\Omega\rangle_{\text{soft only}}$ ← Sudakov form factor.

$= \langle\Omega| T\{W[\infty(-n_B), 0]\, W[0, -\infty(n_A)]\}|\Omega\rangle_{\text{soft only}}$

$$\propto \int \frac{d^4k}{(2\pi)^4} \frac{i(n_B\cdot n_A)}{(n_B\cdot k - i\epsilon)[k^2 - m_g^2 + i\epsilon][n_A\cdot k + i\epsilon]}$$

- dk^- integral first: pick up the residue at a pole (eg. $k^2 - m_g^2 = 0$ pole $\Rightarrow k^- = \frac{\vec{k}_T^2 + m_g^2}{2k^+}$)
- remaining dk^+ integral diverges. if n_A^μ, n_B^μ are light like.

some what different from the UV divergence (hence another term "rapidity divergence")

because it is from an integral within fixed $2k^+k^- = (k^0)^2 - (k^3)^2$. and k_T^2.

The rapidity divergence originates from extending the region of (path) integration from the soft region to all momenta.

a region with $k^+ \gg k_T \gg k^-$ or $k^- \gg k_T \gg k^+$: soft approximation is not good.

Analogy: soft gluon eff. theory $\longleftrightarrow$ non-rela fermion eff. theory

$\left\{\begin{matrix}\text{rapidity} \\ \ln(\mu^2/k_T^2)\end{matrix}\right\}$ divergence $\Longleftrightarrow$ linearly divergent mass correction

$\boxed{\text{non-light like } v_A^\mu, v_B^\mu}$ a gauge-invariant way to regularize this new divergence.

choose $v_A^\mu = (1, -e^{-2\eta_A}, \vec{0}_T)$, $v_B^\mu = (-e^{2\eta_B}, 1, \vec{0}_T)$ } slightly space-like (for Drell-Yan)

(η_A : positive large. η_B : negative large.)

$$\boxed{S(\vec{b}_T; v^\mu_{A,B}; \mu) := \langle\Omega| T\left\{\frac{1}{N_c}\mathrm{Tr}\Big(\overline{W}[0,-\infty(v_A)]\, W[0,-\infty(v_B)]\; \overline{W}[\vec{b}_T, \vec{b}_T-\infty(v_B)]\, W[\vec{b}_T,\vec{b}_T-\infty(v_A)]\Big)\right\}|\Omega\rangle}$$

dim. reg. + $\overline{MS}$ on k_T

soft factor.

At 1-loop

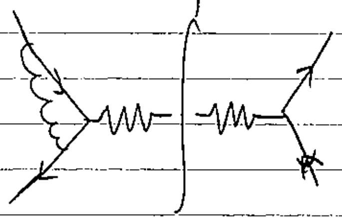
$$\boxed{S = 1 - \frac{\alpha_s}{2\pi} C_2 \ln(\mu^2 b_T^2)(\eta_A-\eta_B)}$$ ← finite, well-defined.

$\left(\frac{\partial(\ln S)}{\partial \ln(\mu^2)} \simeq -\frac{\alpha_s}{2\pi} C_2 (\eta_A-\eta_B) \right.$ is called the cusp anomalous dimension. $)$

$$(\eta_A-\eta_B) = \frac{1}{2}\ln\left(\frac{(v_A\cdot v_B)^2}{v_A^2\, v_B^2}\right)$$

The collinear factor : should not include the soft contributions.

In the 1-loop virtual correction, for example,



$$\int \frac{d^2k^\pm}{(2\pi)^2}\frac{d^n k_T}{(2\pi)^n}\mu^{2-n}\left\{\frac{\bar{v}(\hat{p}_B)\gamma^\mu(\hat{\not p}_A+\not k)\not v_B\, u(\hat{p}_A)}{[k\cdot v_B-i\varepsilon][k^2+i\varepsilon][(\hat{p}_A+k)^2+i\varepsilon]} - \frac{(\bar{v}\gamma^\mu u)(v_B\cdot v_A)}{[k\cdot v_B-i\varepsilon][k^2+i\varepsilon][k\cdot v_A+i\varepsilon]}\right\}$$

subtract the soft contribution

⇒ The integrand "vanishes" when k^μ is in the soft region.

$$J_A(x^-,\vec{b}_T; v_A; \mu) := \frac{\langle h_A(\vec{p}_A)|\,\overline{\psi}(0)\, W[0,-\infty(v_B)]\,\overline{W}[(x^-,\vec{b}_T),(x^-,\vec{b}_T)-\infty(v_B)]\,\psi(x^-,\vec{b}_T)\,|h_A(\vec{p}_A)\rangle}{S(\vec{b}_T; v_A, v_B; \mu)}$$

$$J_B(x^+,\vec{b}_T; v_B; \mu) := \frac{\langle h_B(\vec{p}_B)|\big(\overline{W}[0,-\infty(v_A)]\,\psi(0)\big)\big(\overline{\psi}(x^+,\vec{b}_T)\, W[(x^+,\vec{b}_T),(x^+,\vec{b}_T)-\infty(v_A)]\big)|h_B(\vec{p}_B)\rangle}{S(\vec{b}_T; v_A, v_B; \mu)}$$

$$H(x^+,x^-,\vec{b}_T;\mu) := \frac{\int d\Pi_X\, \langle h_A h_B|J^\mu(0)|X\rangle\langle X|J^\nu(x)|h_A h_B\rangle}{J_A(x^-,\vec{b}_T;v_A;\mu)\, J_B(x^+,\vec{b}_T;v_B,\mu)\, S(\vec{b}_T;v_{A,B};\mu)}.$$

$$(*)^{\mu\nu} = \int d^2x^{\pm} d^2\vec{b}_T \; e^{iq\cdot x} \; S(\vec{b}_T)\, J_A(x^-, \vec{b}_T)\, J_B(x^+, \vec{b}_T)\, H(x^+ x^-, \vec{b}_T)$$

TMD-factorization

$$= \int d^2x^{\pm} \int \frac{d^2\vec{k}_S}{(2\pi)^2} \frac{d^2\vec{k}_A}{(2\pi)^2} \frac{d^2\vec{k}_B}{(2\pi)^2} \frac{d^2\vec{k}_H}{(2\pi)^2} (2\pi)^2 \delta^2(\vec{q}_T - \vec{k}_S - \vec{k}_A - \vec{k}_B - \vec{k}_H)\, \tilde{S}(\vec{k}_S)\, \tilde{J}_A(x^-, \vec{k}_A)\, \tilde{J}_B(x^+, \vec{k}_B)\, \tilde{H}(x^\pm, \vec{k}_H)$$

- v_A, v_B-dependence should cancel between S and $J_{A,B}$.
  but we still need them to define S, J_A and J_B.
- The Drell-Yan $\frac{d\sigma_{DY}}{d^4q}$ data at small $\vec{q}_T$ can be used to constrain $\vec{k}_A$-dependence of $\tilde{J}_A(x^-, \vec{k}_A)$
  ($\vec{b}_T$-dep.) ($J_A(x^-, \vec{b}_T)$)

similar to the $\left\{\begin{matrix}\text{renormalization} \\ \text{factorization}\end{matrix}\right\}$ scale