A Quick Course on String Compactification mainly to $\mathbb{R}^{3,1}$

§1 Compactification in 10D SUGRA Language

(Ref: GSW §13–16 ~~for §2.1–13.~~)

~~String vacua~~ constructed via $\begin{pmatrix}\text{free boson}\\ \mathbb{R}^{3,1}\end{pmatrix} \otimes \begin{pmatrix}\text{unitary compact}\\ \text{CFT w/ modular inv.}\end{pmatrix}$

are all qualified for the descript'n of this universe.

$(c,\tilde{c}) = (9,9)$ TypeII
$= (22,9)$ Het

"compact" $\overset{\text{def}}{\Leftrightarrow}$ the spectrum of conformal weights is discrete.

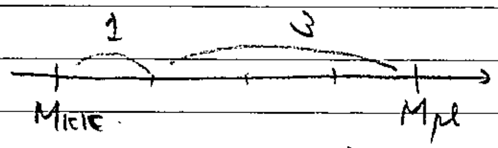
Both in TypeII and Het string,

$$S \supset \frac{2\pi}{(2\pi)^8(\alpha')^4}\int d^{10}x\sqrt{-g}\,e^{-2\phi}R \Rightarrow \frac{2\pi\langle \mathrm{vol}_6\rangle}{(2\pi)^8(\alpha')^4(g_s)^2}\int d^4x\sqrt{-g}\,R.$$

$$\boxed{\frac{1}{16\pi G_N} = \frac{M_{pl}^2}{2}} \qquad \boxed{M_{pl} \simeq 2.4\times 10^{18}\,\mathrm{GeV}.}$$

Roughly speaking, $M_{KK} := \frac{1}{\langle \mathrm{vol}_6\rangle^{1/6}}$ $\quad M_s := \frac{1}{l_s} := \frac{1}{2\pi\sqrt{\alpha'}}$

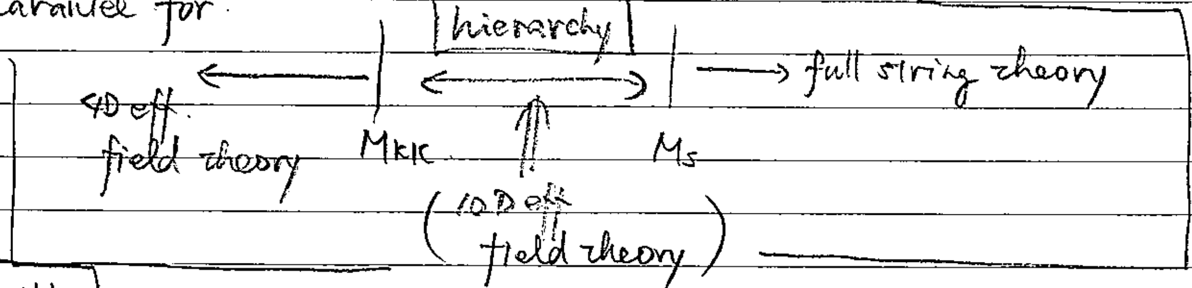
$$\boxed{4\pi M_s^8 = M_{KK}^6 M_{pl}^2 g_s^2}$$



$g_s^2/(4\pi)$ ignored.

so $\boxed{M_{KK} \lesssim M_s \lesssim M_{pl}.}$

No guarantee for $\boxed{\text{hierarchy}}$



$\boxed{\text{possible}}$

- ~~$M_s \sim M_{KK} \sim M_{pl}$ and $g_s \sim \mathcal{O}(1)$~~
- $(M_s \sim M_{KK}) \ll M_{pl}$ and $g_s \ll 1$.
- $M_s < M_{KK}$ because $g_s \ll 1$.

no 10D SUGRA approximat'n

$\boxed{\text{always } M_s \lesssim M_{pl}.}$

- hybrid: ($^{\exists}(4 < d < 10)$ d-dim (SU)GRA approximat'n exists)
- $M_{KK} \ll M_s$: 10D SUGRA approximat'n exists.

examples

- S^1-compactificatn @ self dual radius ($R=\sqrt{\alpha'}$)
  - $\rightarrow$ $SU(2)$ gauge field in $(8+1)$-dim / $(24+1)$-dim space-time effective theory
  - $\rightarrow$ does not make sense to retain KK modes w/o winding in the $(9+1)$-dim effective field theory or vice versa.
- $c=\tilde{c}=1$ bosonic string $\exists$ 3 isolated CFT's not interpreted as either S^1 or $S^1/\mathbb{Z}_2$. (Polchinski Chapt. 8)
- more

space-time dimension : define by [harmonic mode counting / CFT central charge] ?

$\boxed{\text{The } M_{KK} \ll M_s \text{ cases}}$

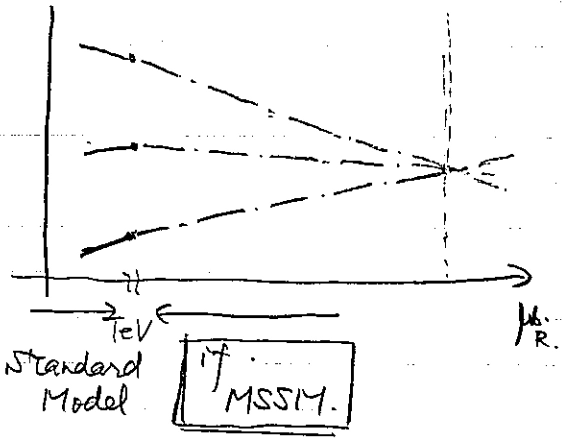
- easier to construct and to extract their consequences, often systematically
- motivated if you believe in Supersymmetric Grand Unification.
  set. $M_{KK} \simeq M_{GUT}$ $((1\sim3)\times10^{16}\,\mathrm{GeV})$ $\Leftarrow$ SUSY gauge coupling unification.

Ref.
- Witten Nucl. Phys. B258 ('85) 75.
- Witten Nucl. Phys. B268 ('85) 79.
- pheno literatures : ask me directly.

§1.1 Bottom-up Motivations for Low-energy SUSY

∘ gauge coupling unificatn.

is observed, if we assume that the spectrum of the SM is supersymmetrized soon above the TeV scale.



$$\frac{\partial}{\partial \ln\mu}\left(\frac{1}{\alpha_i(\mu)}\right) = \frac{b_i}{2\pi}$$ ← depends on the matter contents.

at 1-loop

↑ more solid

∘ a stable particle for dark matter.

The dark matter particle needs to be (almost) stable, and has to have been produced by the ammount ρ_{DM} consistent with cosmological measuremente today.

☆ 100 GeV — 100 TeV stable particle thermal relic.
(Griest Kamionkowski '90 PRL)

☆ axion (a solution to the strong CP problem) is an alternative.

↓ less solid.

∘ Naturalness problem (a strategic question)

Higgs potential and inflaton potential.

§1.2. Special Holonomy Mfds and Partial Supersymmetry Breaking

Remember in $D=4$ $\mathcal{N}=1$ SUSY gauge theories...

$$\delta\lambda_\alpha = -\frac{1}{4} F_{\mu\nu}(\sigma^{\mu\nu})_\alpha{}^\beta \xi_\beta + D\xi_\alpha$$

$$\delta\bar{\lambda}_{\dot{\alpha}} = -\frac{1}{4} F_{\mu\nu}(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} \bar{\xi}^{\dot{\beta}} + D\bar{\xi}^{\dot{\alpha}}$$

if $\langle D\rangle = 0$ and $\langle F_{\mu\nu}\rangle(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} = 0$ but $\langle F_{\mu\nu}\rangle(\sigma^{\mu\nu})_\alpha{}^\beta \neq 0$.

(instanton configuration)

then half of the 4 SUSY charges remain unbroken.

- ☆ Euclidean 4-dim : $SO(4) \simeq SU(2) \times SU(2)$
  - ξ_α $\bar{\xi}_{\dot{\alpha}}$ pseudo real
  - $2 + 2$ (symplectic)
  - $= 4$ SUSY charges
- ☆ $SO(4)$ rotational symmetry is lost at the same time

Analogy :

$\mathbb{R}^{9,1}$ replaced by $M \times \mathbb{R}^{3,1}$

M: 6-$\dim_{\mathbb{R}}$ mfd.

non-trivial background (= vac. value) configurat'n of metric & gauge fields on M

may
- break $SO(6)$ symmetry but preserve $SO(3,1)$
- break a part of SUSY but preserve the rest of the SUSY

An example : Heterotic string

10D SUGRA action (α'-expansion)

Polchinski §12.1 (12.1.39) $\left\{\begin{array}{l}\text{bosonic part only} \\ \text{string frame}\end{array}\right\}$

$$\frac{1}{\sqrt{-\det(g_s)}}\mathcal{L} = \frac{1}{2\kappa_{10}^2} e^{-2\phi}\left[R + 4(\partial\phi)^2 - \frac{1}{2}|\tilde{H}_3|^2 - (\cdots)\mathrm{Tr}\left(|F|^2\right)\right]$$

$$+ (\text{fermionic})$$

GSW §13 p.313 + p.323 $\left\{\begin{array}{l}\text{incl. fermionic terms} \\ \text{Einstein frame.}\end{array}\right\}$

bosonic DOF $(g_{MN},\ \phi,\ B_{MN}) + (F^a_{MN})$

massless

fermionic DOF $(\Psi_M,\ \lambda) \quad + (\chi^a)$

56 + 8 on-shell

SUSY transformation. (10D N=1 SUGRA) (GSW p.416 (§15))

$$\begin{cases} \delta\Psi_M = \dfrac{1}{\kappa_{10}}(D_M\eta) + (\cdots)\, e^{-\phi/2}\, (\Gamma\text{'s})_M{}^{NPQ}\, \eta\, \tilde{H}_{NPQ} + \cdots \\ \delta\lambda = -\dfrac{1}{2\sqrt{2}}\Gamma^M(\partial_M\phi)\,\eta + (\cdots)\, e^{-\phi/2}\, \Gamma^{MNP}\tilde{H}_{MNP}\,\eta + \cdots \\ \delta\chi^a = -\dfrac{1}{4}(\cdots)\, e^{-\phi/4}\, \Gamma^{MN}F^a_{MN}\,\eta + \cdots \end{cases}$$

$\langle$ fermionic terms , higher-order terms in α' ignored. $\rangle$

(Ref: Polchinski appendix B)

η : SUSY transformation parameter in $\underline{16}$-repr. of $SO(9,1)$

Majorana Weyl spinor

In the irr. decomposition

$$SO(9,1) \longrightarrow SO(3,1) \times SO(6)^{\sim SU(4)}$$

$$\boxed{\begin{array}{lll} 16 & \longrightarrow & [(2_L,1) \otimes 4] \\ & & \oplus [(1,2_R) \otimes \bar{4}] \quad \text{CPT} \\ \Big(16' & \longrightarrow & [(2_L,1) \otimes \bar{4}] \\ & & \oplus [(1,2_R) \otimes 4] \quad \text{CPT} \Big) \end{array}}$$

$$SO(9,1) \longrightarrow SO(5,1) \times \left(SU(2)\times SU(2) \simeq SO(4)\right)$$

$$\boxed{\begin{array}{lll} 16 & \longrightarrow & [4 \otimes (2,1)] \ \text{CPT} \\ & & \oplus [4' \otimes (1,2)] \ \text{CPT} \\ \Big(16' & \longrightarrow & [4 \otimes (1,2)] \\ & & \oplus [4' \otimes (2,1)] \Big) \end{array}}$$

Via the separation of variables $\eta(x,y) \Rightarrow \varepsilon_\alpha(x)\otimes\eta_0(y) + \bar{\varepsilon}_{\dot\alpha}(x)\otimes\bar{\eta}_0(y)$

$\left(10D \longrightarrow \mathbb{R}^{3,1} \times (6D \text{ internal}) \right)$

$$\Psi_\mu(x,y) \longrightarrow \psi_{\mu\alpha}(x)\,\eta_0(y) + \bar{\psi}_{\mu,\dot\alpha}(x)\,\bar{\eta}_0(y)$$

$$(\delta\psi_{\mu\alpha}(x))\,\eta_0(y) = \frac{1}{K_{10}}(D_\mu\varepsilon_\alpha)\,\eta_0(y) + \cdots \quad \text{as in 4D SUGRA.}$$

$$\Psi_m(x,y) \longrightarrow \varepsilon_\alpha(x)\cdot\psi_m(y) + \bar{\varepsilon}_{\dot\alpha}(x)\,\bar{\psi}_m(y)$$

$$\varepsilon_\alpha(x)\,(\delta\psi_m(y)) = \frac{1}{K_{10}}\,\varepsilon_\alpha(x)\,(D_m\eta_0)(y) + \cdots$$

$\Rightarrow$ independent covariantly constant spinors $(D_m\eta_0^i) = 0$ on M (6$D_{\mathbb{R}}$ internal mfd), $i = 1, 2, \cdots, N$.

give rise to $\eta(x,y) \Rightarrow \varepsilon_\alpha^i(x)\,\eta_0^i(y) + \text{c.c.}$

N extended SUSY in $(3+1)$-dim. $\mathbb{R}^{3,1}$

Special holonomy manifold M_n

Levi-Civita connection takes its value in $\mathfrak{g} \subsetneq \mathfrak{so}(n)$.
(Christoffel symbol)

$$0 = D_m \eta = \left(\partial_m \delta^\alpha{}_\beta + \omega_m^{ab} (S^{ab})^\alpha{}_\beta\right) \eta^\beta = 0.$$

α, β: spinor index
$a, b, = 1, \cdots, n$
$m = 1, \cdots, n$.
so, if $\langle \omega_m^{ab}(y) \rangle$ is less than $\mathfrak{so}(n)$ at any point in M_n,
there is a chance to find $\eta_0 \neq 0$ w/ $D_m \eta_0 = 0$

☆ Kähler manifold ($\Rightarrow$ cpx mfd by def)

$U(n)$ holonomy $\subset SO(2n)$ $\qquad$ $\Gamma^\alpha_{\beta\gamma} \neq 0$ $\Gamma^{\bar\alpha}_{\bar\beta\bar\gamma} \neq 0$
but all other Γ^m_{nl}'s vanish.
$\Uparrow$
(cf. Wess-Bagger appendix C)

in the vector repr. of $SO(2n)$

$$\begin{pmatrix} \boxed{U(n)} & 0 \\ 0 & \boxed{U(n)}^{cc} \end{pmatrix} \begin{pmatrix} \text{hol} \\ \overline{\text{hol}} \end{pmatrix}$$

☆ Calabi-Yau cpx n-fold (real $2n$ dimension)

$SU(n) \subset U(n) \subset SO(2n)$

$$\begin{array}{c} SU(4) \\ \wr\!\wr \\ SO(6) \longrightarrow U(3) = SU(3) \times U(1) \end{array}$$

$\underline{\underline{n=3}}$ the spinor repr. of $SO(6) \simeq SU(4)$ $\qquad 4 \oplus \bar{4} \longrightarrow (1 \oplus 3) \oplus (1 \oplus \bar{3})$

$$\left(|\uparrow\uparrow\uparrow\rangle, |\uparrow\downarrow\downarrow\rangle, |\downarrow\uparrow\downarrow\rangle, |\downarrow\downarrow\uparrow\rangle\right) \oplus \left(|\downarrow\downarrow\downarrow\rangle, |\downarrow\uparrow\uparrow\rangle\ |\uparrow\downarrow\uparrow\rangle\ |\uparrow\uparrow\downarrow\rangle\right)$$

$$\underbrace{(\wedge^3 TM)^{1/2} \oplus (\wedge^3 TM)^{-1/2} \otimes (TM)}_{= S_+} \qquad \underbrace{(\wedge^3 TM)^{-1/2} \oplus (\wedge^3 TM)^{+1/2} \otimes (T^*M)}_{= S_-}$$

S_+ (rank=4 spinor bdle over M)

$SU(3)$ holonomy $\Leftrightarrow$ $D_m \eta_0 = (\partial_m + \omega_m)\, \eta_0$ $\qquad \overset{(4\times 4)}{\cup}$
$\hookrightarrow$ value in (3×3) part
$\Leftrightarrow$ $(\wedge^3 TM)$ is a trivial bundle.

$\underline{n=2}$ (complex surface)

The spinor repr. of $SO(4) \simeq SU(2)\times SU(2)$ is $(2,1)\oplus(1,2)$.

$$(|\uparrow\uparrow\rangle,\ |\downarrow\downarrow\rangle) \oplus (|\uparrow\downarrow\rangle,\ |\downarrow\uparrow\rangle)$$

$$S_+ = (\Lambda^2 TM)^{1/2}\oplus(\Lambda^2 TM)^{-1/2} \qquad S_- = (\Lambda^2 TM)^{-1/2}\otimes TM = (\Lambda^2 TM)^{1/2}\otimes(T^*M)$$

If $\langle \omega_m \rangle$ the spin connect'n is with in

$\underset{\sim}{SU(2)} \subset U(2) \subset SO(4)$.
↳ one of the two $SU(2)\times SU(2) \simeq SO(4)$

then $D_m \eta_0 = 0$ condition becomes $d\eta_0 = 0$
in the other $SU(2)$ doublet.

$\underline{n=1}$: T^2, (complex torus) $\Rightarrow$ transl.-inv. metric $\Rightarrow \langle\omega_m\rangle = 0$
All the SUSY charges remain unbroken.

$$\begin{cases} \text{Het}/T^2 \Rightarrow 16 \text{ SUSY charges in } (7+1)\text{-dim eff. theory} \\ \text{Type II}/T^2 \Rightarrow 32 \text{ SUSY charges in } (7+1)\text{-dim eff. theory} \end{cases}$$

$\underline{n=2}$: K3 surface (cpx surface M w/ trivial$(\Lambda^2 TM)$ & $h^{1,0}=0$)

$$\left[\begin{array}{ll} \text{Het}/K3 \Rightarrow (0,1) \text{ SUSY} & \\ \text{Type IIA}/K3 \Rightarrow (1,1) \text{ SUSY} & \text{in } (5+1)\text{-dim eff. theory.} \\ \text{Type IIB}/K3 \Rightarrow (\underset{\sim}{0},\underset{\approx}{2}) \text{ SUSY} & \end{array}\right.$$

[SUSY charges in 4 of $SO(5,1)$]   # [SUSY charges in 4' of $SO(5,1)$]

If $h^{1,0} > 0 \Rightarrow D_m\eta = 0$ w/ $\eta \in (T^*M \simeq S_-)$. More SUSY charges in $(5+1)$-dim. theory.

[$M = T^4 \Rightarrow h^{1,0}(M) = 2$, $\mathcal{N} = (1,1), (2,2), (2,2)$ resp.]

$\underline{\underline{n=3}}$ Calabi-Yau 3-fold $\left(\begin{array}{l}\text{cpx 3-dim. Kähler } M \text{ w/} \\ \quad \text{trivial } (\Lambda^3 TM) \ \& \ h^{1,0}=0\end{array}\right)$

$$\left[\begin{array}{ll} \text{Het}/CY_3 \Rightarrow N=1 \text{ SUSY} & \\ \text{IIA}/CY_3 \Rightarrow N=2 \text{ SUSY} & \text{in } (3+1)\text{-dim effective theory.} \\ \text{IIB}/CY_3 \Rightarrow N=2 \text{ SUSY} & \end{array}\right.$$

If $h^{1,0}>0$ then the eff. theory has $\overset{h^{1,0}(\times 2)}{\text{more}}$ SUSY charges.

$$\left[\begin{array}{lll} M = K3 \times T^2 & \Rightarrow h^{1,0}=1 & N=1+1 \quad (2+2) \\ M = T^6 & \Rightarrow h^{1,0}=3. & N=1+3 \quad (2+6) \end{array}\right.$$

☆ G_2-holonomy : M is a real mfd of dim $=7$.

$SO(7)$ has just one spinor repr. $\underline{\underline{8}}$. (see E. Witten hep-th/0108165 pp. 7–8)

$\mathrm{Res}^{SO(7)}_{G_2}(\underline{\underline{8}}) = \underline{\underline{7}} \oplus \underline{\underline{1}}$.

$$\boxed{(11D \text{ SUGRA})/M \Rightarrow N=1 \text{ SUSY charges in } (3+1)\text{-dim eff. theory}}$$

$\left(32 \times \frac{1}{8} = 4\right)$

roughly speaking $\underset{\text{diagonal}}{SU(2)} \times SU(2) \subseteq G_2$ holonomy

$SO(3) \times \underbrace{SU(2) \times SU(2)}_{SO(4)} \subset SO(7)$

spinor $\underline{\underline{8}} = (\underline{\underline{2}} \otimes \underline{\underline{2}} \otimes 1)$
$\oplus (\underline{\underline{2}} \otimes 1 \otimes \underline{\underline{2}})$

non-trivial holonomy

$1 \oplus \underline{\underline{3}}$ under the diagonal $SU(2) \subset SO(3) \times SU(2)$.

non-trivial holonomy

$\Downarrow$

trivial holonomy

cov. const spinor lives here.

Bogomolov – Beauville classificat'n thm

M: compact Kähler mfd w/ Ricci flat metric.

${}^{\exists}M'$ finite cover of M such that

$$M' = (\text{cpx torus}) \times \begin{pmatrix}\text{simply connected}\\ SU(n)\text{-holonomy mfds}\end{pmatrix} \times \begin{pmatrix}\text{simply connected}\\ Sp(r)\text{-holonomy mfds}\end{pmatrix}$$

$\Downarrow$ $\boxed{h^{1,0} = \cdots h^{n-1,0} = 0}$ $\Downarrow$ $\boxed{\begin{array}{l} h^{1,0} = h^{3,0} = \cdots = 0 \\ h^{2,0} = h^{4,0} = \cdots = h^{2r,0} = 1 \end{array}}$

So, if $\dim_{\mathbb{C}} M = 3$ then M' can only be

CY_3, $K3 \times T^2$, T^6.

If $\dim_{\mathbb{C}} M = 2$: $M' = K3$ or T^4

If $\dim_{\mathbb{C}} M = 4$: $M' = K3^{[2]}$, $CY_3 \times T^2$, $K3 \times K3$, $K3 \times T^4$, T^8

Lie algebra $\mathfrak{so}(8)$, $\mathfrak{so}(7)$, $\mathfrak{g}_2$

$\mathfrak{so}(8)$. Cartan diagonal basis. $\left\{ X \in \underset{\mathbb{C}\text{-val matrix}}{8\times 8} \;\middle|\; \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}[X] + [X]^T\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 0 \right\}$

$\Leftrightarrow \left\{ X \in (\;\vdots\;) \;\middle|\; X = \begin{bmatrix} A & B \\ C & -A^T \end{bmatrix} \; B, C \text{ anti-sym} \right\}$

Generators (basis) of $\mathfrak{h}$: $H_{i=1,2,3,4} \Leftrightarrow A = \mathrm{diag}(0 \cdots \underset{\substack{\uparrow \\ i}}{1} \cdots 0)$

Weights $\{L_i\}_{i=1\sim4}$ the dual basis of $H_{i=1\sim4}$.

$\Rightarrow$ The roots of $\mathfrak{so}(8)$: $\{\pm(L_i - L_j)_{i<j}, \pm(L_i + L_j)_{i<j}\}$ total 24.

The Cartan subalg of $\mathfrak{so}(7)$: $\underset{SO(4)\times SO(3)}{\mathrm{Span}\{H_1, H_2, H_3\}}$.

$\left(\begin{array}{l} \text{vector repr } (\pm L_{1,2,3,4}) \Rightarrow \text{vector } (\pm L_{1,2,3}, 0) \\ \qquad\qquad + (\text{neutral} \oplus 1) \\ \text{spinor repr's both} \Rightarrow \frac{1}{2}(\Sigma_i \varepsilon_i L_i) \text{ spinor} \end{array}\right)$

The Cartan subalg of $\mathfrak{g}_2$: $\mathrm{Span}\{\underbrace{\tfrac{1}{2}(H_1 - H_2) + H_3}_{= X_1}, \underbrace{(H_1 + H_2)}_{= X_2}\}$

$[SO(4) \simeq SU(2) \times SU(2)] \times SO(3)$

$(H_1 + H_2)$, $(H_1 - H_2)$ $\quad H_3$

diagonal

rank = 2 $\mathfrak{g}_2$. as in p(8)

$\begin{pmatrix} \langle X_1, L_1 - L_2 \rangle \\ \langle X_2, L_1 - L_2 \rangle \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ also $L_3 \pm L_4$, $\quad \begin{pmatrix} \langle X_1, L_2 + L_3 \rangle \\ \langle X_2, L_2 + L_3 \rangle \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$ also $L_1 \pm L_4$, $\quad \begin{pmatrix} \langle X_1, L_1 - L_3 \rangle \\ \langle X_2, L_1 - L_3 \rangle \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$ also $L_2 \pm L_4$

$\begin{pmatrix} \langle X_1, L_1 + L_3 \rangle \\ \langle X_2, L_1 + L_3 \rangle \end{pmatrix} = \begin{pmatrix} 3/2 \\ 1 \end{pmatrix}$, $\quad \begin{pmatrix} \langle X_1, L_1 + L_2 \rangle \\ \langle X_2, L_1 + L_2 \rangle \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, $\quad \begin{pmatrix} \langle X_1, L_2 - L_3 \rangle \\ \langle X_2, L_2 - L_3 \rangle \end{pmatrix} = \begin{pmatrix} -3/2 \\ 1 \end{pmatrix}$ PLUS