## §1.3. Low-energy Spectrum of Closed String

IIA 10D SUGRA:

$$(g_{MN}, \phi, B_{MN}; C^{(1)}, C^{(3)})_{boson} + (\text{fermion})$$

We just need to work out bosonic DOF's
in a supersymmetric compactificat'n. (SUSY determines fermionic spectrum.)

Consider IIA/$M_6 = CY_3$

$$\begin{cases} M_6: \text{ cpx str \& Kähler metric given} \iff \langle g_{mn}(y)\rangle \neq 0 \text{ given.} \\ \langle\phi\rangle = \text{const. over } M_6. \quad \langle B_{mn}(y)\rangle \neq 0 \text{ harmonic form.} \\ \langle C^{(1)}\rangle = \langle C^{(3)}\rangle = 0. \end{cases}$$

a sufficient condition for $N=2$ SUSY in $(3+1)$-dim. effective theory.

$$(\text{field})(x,y) = \langle\text{field}\rangle(y) + (\text{field fluctuation})(x,y)$$
$$\| $$
$$\sum_i (\text{Low-energy field})_i(x)\,(\text{wave fcn})_i(y).$$

separat'n of variables

In many cases (with an appropriate gauge fixing)
the equations on the wave functions after the separation of variables is the Laplace equation.

$$\begin{aligned}
(\delta\phi)(x,y) &= \phi(x)\cdot 1 + (\text{KK-modes}) \\
(\delta B_{\mu\nu})(x,y) &= b_{\mu\nu}(x)\cdot 1 + (\text{KK-modes}) \\
(\delta B_{\mu n})(x,y) &= (\text{KK-modes}) \quad \text{no harmonic 1-form on } M \ (h^{1,0}(M)=0) \\
(\delta B_{mn})(x,y) &= \sum_{a=1}^{h^{1,1}(M)} \left((\omega^a)_{mn}(y)\,\beta^a(x)\right) + (\text{KK-modes})
\end{aligned}$$

$$\left(H^{1,1}(M;\mathbb{R}) = \mathrm{Span}_{\mathbb{R}}\{\omega^a \mid a=1,2,\dots,h^{1,1}(M)\}\right)$$

$$(\delta C^{(1)}_{\mu}) = c_\mu(x)\cdot 1 + (\text{KK-modes})$$

$$(\delta C^{(1)}_{m}) = (\text{KK-modes}) \longleftarrow (\text{no harmonic 1-form on } M.)$$

$$(\delta C^{(3)}_{\mu\nu\lambda}) = \text{non dynamical} + (\text{KK-modes})$$

$$(\delta C^{(3)}_{\mu\nu m}) = (\text{KK-modes})$$

$$(\delta C^{(3)}_{\mu mn}) = \sum_{a=1}^{h^{1,1}} \left(A_\mu{}^a(x)(\omega^a{}_{mn})(y)\right) + (\text{KK-modes})$$

$$(\delta C^{(3)}_{mn\ell}) = \sum_{i=1}^{b_3(M)} \left(C_i(x)\,(\psi_i)_{mn\ell}(y)\right) + (\text{KK-modes})$$

$$(\delta g_{\mu\nu}) = (\mathbb{R}^{3,1}\text{ graviton}) + (\text{KK-modes})$$

$$(\delta g_{\mu n}) = (\text{KK-modes})$$

$$(\delta g_{\alpha\bar\beta}) = \sum_{a=1}^{h^{1,1}} \left(\alpha^a(x)\,\omega^a_{\alpha\bar\beta}(y)\right) + (\text{KK-modes})$$

$$(\delta g_{\bar\alpha\bar\beta}) = \sum_{i=1}^{h^{2,1}(M)} \left(z_i(x)\ \langle g_{\bar\alpha\delta}\rangle_{(y)}(J_i)^\delta{}_{\bar\beta}(y)\right) + (\text{KK-modes})$$

$\{J_i\}$ : basis of $H^1(M;TM)$

$\Updownarrow$ Calabi-Yau.

$\{\psi_i\}$ basis of $H^{2,1}(M;\mathbb{C})$ $\left(\begin{matrix}3 \simeq \Lambda^2\bar{3}\\ \text{in } SU(3)\end{matrix}\right)$

via $\boxed{(d\bar z^{\bar\beta} J^\delta{}_{\bar\beta})\frac{\partial}{\partial z^\delta}} \longmapsto \Omega_{\alpha\gamma\delta}J^\delta_{\bar\beta} = \psi_{\alpha\gamma\bar\beta}$

$\boxed{(\psi_{\alpha\gamma\bar\beta}\ dz^\alpha\wedge dz^\gamma\wedge d\bar z^{\bar\beta})}$

the tangent space of
the moduli space of complex structure of $M$.

(K. Kodaira { "Cpx Mfd" [Iwanami Pub. Co].
Lecture note @ U.Tokyo
「複素多様体と複素構造の変形」)

[to go beyond the DOF counting ...]

see

(Refs. E. Witten Phys. Lett. B155 ('85) 151. Het

Louis et al. hep-th/0312232 ) IIB

hep-th/0403067 )

hep-th/0412277 ) IIA.

also [Candelas de la Ossa Nucl. Phys. B355 ('91) 455.]

## SUMMARY

$\mathrm{IIA}/(M_6 = CY_3)$

| massless fields in $\mathbb{R}^{3,1}$ | $\mathbb{R}^{3,1}$ $N=2$ SUSY |
|---|---|
| (graviton) . $C_\mu$ (from $C^{(1)}$) | sugra mult. |
| $h^{1,1} \times \left( A_\mu^a \text{ (from } C^{(3)}), \ \alpha^a \text{ (metric)}, \ \beta^a \ (B^{(2)}) \right)$ | vector mult |
| $h^{2,1} \times \left( z^i \text{ (from } \delta g_{\bar\alpha\bar\beta}), \ C_i, C_{i'} \text{ (from } C^{(3)}) \right)$ | hyper mult |
| $(\phi, \ b_{\mu\nu}, \ C_{i=0}, \ C_{i'=0})$ | linear/tensor mult. ($\simeq$ hyper mult.) |

$\boxed{\begin{array}{l} h^{1,1}(M_6) \ \text{ vector multiplets } \ U(1) \\ (h^{2,1}(M_6)+1) \ \text{ hyper multiplets} \end{array}}$

mirror symmetry.

$\mathrm{IIB}/(W_6 = CY_3)$ then

$$\left( g_{MN}, \ \phi, \ B_{MN} \ ; \ C^{(0)}, \ C^{(2)}, \ C^{(4)} \right) \Longrightarrow$$

$\mathbb{R}^{3,1}$ $N=2$ SUSY

$\boxed{\begin{array}{l} 1 \ \text{ SUGRA multiplet} \\ h^{2,1}(W_6) \ \text{ vector multiplets } U(1) \\ h^{1,1}(W_6)+1 \ \text{ hyper multiplets} \end{array}}$

Hodge diamond

$$\begin{array}{ccccccc} & & & (h^{0,0}=1) & & & \\ & & 0 & & 0 & & \\ & 0 & & \boxed{h^{1,1}} & & 0 & \\ (h^{3,0}=1) & & \boxed{h^{2,1}} & & (h^{1,2}=h^{2,1}) & & (h^{0,3}=1) \\ & 0 & & (h^{2,2}=h^{1,1}) & & 0 & \\ & & 0 & & 0 & & \\ & & & (h^{3,3}=1) & & & \end{array}$$

$\Uparrow$

$(CY_3)$

$$h^{p,q} := \dim_{\mathbb{C}} \left( H^{p,q}(M;\mathbb{C}) \right)$$

$\boxed{\begin{array}{l} H^{p,q}(M;\mathbb{C}) \simeq H^q(M; \wedge^p T^* M) \\ \quad \text{if } M : \text{Kähler mfd.} \end{array}}$

- $h^{p,q} = h^{q,p}$ (complex conjugation)

if $CY_n$ : $\begin{cases} h^{n,0} = 1 & \Leftarrow (\wedge^n T^* M \text{ is trivial}) \\ h^{p,q} = h^{n-p,n-q} & \Leftarrow (\text{Serre duality}) \\ h^{n,q} = h^{0,q} & \end{cases}$

Het 10D SUGRA

( $g_{MN}$, $\phi$, $B_{MN}$ ; $A^a_M$ )

$\longrightarrow$ gauge field in $SO(32)$ or $E_8 \times E_8$.

A class of
$N=1$ SUSY preserving background configurat'n

$$\boxed{\begin{cases} M_6 = CY_3 & (\text{cpx str \& Kähler metric given.}) \\ \langle \phi \rangle = \text{const.} & \langle B^{(2)} \rangle \\ F^a_{\alpha\beta} = F^a_{\bar\alpha\bar\beta} = 0. & F^a_{\alpha\bar\beta}\, g^{\alpha\bar\beta} = 0 \quad \text{on } M_6. \end{cases}} \quad (*)$$

sufficient condition for $N=1$. (not necessary)

$(*)$ needs to be subject to a constraint. $\mathrm{tr}_{6\times6}\left[\left(\frac{R}{2\pi}\right)^2\right] - T_R^{-1}\mathrm{tr}_R\left[\left(\frac{F}{2\pi}\right)^2\right] = 0$ on $M_6$. $\left(\begin{matrix}\text{or}\\ = 4\\ \text{PD[2cycle]}\end{matrix}\right)$

(both anti-Hermitian)

$$0 \longrightarrow \mathrm{Hom}(H_2(M_6;\mathbb{Z}), \mathbb{R}/\mathbb{Z}) \longrightarrow \underset{\substack{\uparrow \\ \text{inter-patch} \\ \text{gluing}}}{\underset{(R^2-F^2)}{(\cdot)}} \longrightarrow H^3(M_6;\mathbb{Z})$$

$e^{2\pi i\int_\Sigma\left(\frac{B}{\ell_s^2}\right)}$

$\left[\begin{matrix}\text{or}\\ H^3(M_6 \setminus (2\text{-cycle});\mathbb{Z})\end{matrix}\right] \Rightarrow \boxed{4\, c_2(TM_{hol}) + 2\, T_R^{-1} ch_2(V_R) = \begin{cases} 0 \\ \text{or } 4\times \text{PD[2cycle]}\end{cases}}$ (repr)

## The spectrum in $\mathbb{R}^{3,1}$ ($N=1$ SUSY)

- ◎ SUGRA multiplet ( $g_{\mu\nu}$ )
- Kähler (moduli) → ◎ $h^{1,1}$ chiral multiplet ( $(\alpha^a(x) + i\beta^a(x))$ from $(\delta g_{\alpha\bar\beta}, \delta B_{\alpha\bar\beta}) \propto \omega^a_{\alpha\bar\beta}$ )
- cpx str (moduli) → ◎ $h^{2,1}$ 〃 〃 ( $z^i$ from $(\delta g_{\bar\alpha\bar\beta}) \propto \langle g_{\bar\alpha\delta}\rangle J^{\delta}{}_{\bar\beta}$ )
- universal → ◎ 1 chiral multiplet ( $\phi_{4D}$, $b_{\mu\nu}$ ) ( $\phi_{4D} = \delta\phi_{10D}$ + lin comb of $\alpha^a$ )

and ....

if $G_{str} \times H \subset G$ and $\mathrm{Res}^G_{G_{str}\times H}(\mathrm{adj}) = (\mathrm{adj}, 1) + (1, \mathrm{adj}) + \bigoplus_{a\in A} (\rho_a, R_a)$

then.

- ◎ 1 vector multiplet $A_\mu^a$ in $(1, \mathrm{adj})$
- ◎ $H^2(M_6; \rho_a(V))$ chiral multiplets in repr. $R_a$ of $H$
- vect. bdle (moduli) → ◎ $H^2(M_6; \mathrm{adj}(V))$ chiral multiplets neutral under $H$.

**Comment**

- Ricci flat & Kähler $\Longrightarrow$ $\det(TM)$ is trivial
  ($c_1(TM)$ is)
- $\det(TM)$ is trivial & Kähler $\Longrightarrow$ Ricci flat.

Kähler $\Rightarrow$ $R_{\alpha\bar\beta\gamma\bar\delta} = K_{\alpha\gamma\bar\beta\bar\delta} - K_{\alpha\gamma\bar\rho} K_{\bar\beta\bar\delta\sigma} K^{\sigma\bar\rho}$

so...

(Wess Bagger appendix C)

$$\boxed{\begin{array}{c} R_{\gamma\bar\delta} = R^{\beta}{}_{\gamma\beta\bar\delta} = K^{\beta\bar\alpha} R_{\bar\alpha\gamma\beta\bar\delta} = K^{\beta\bar\alpha} R_{\bar\alpha\beta\gamma\bar\delta} = R^{\beta}{}_{\beta\gamma\bar\delta} \\ \text{Ricci flat} \Longleftrightarrow \text{traceless} \end{array}}$$

(related discuss will be found in §3.1)

**Comment**

$M$ : compact Kähler manifold.
If $h^{2,0}(M) = 0$.
$^\exists$ holomorphic embedding $M \longrightarrow \mathbb{P}^N$.

(rough idea)

In the long exact sequence associated with $0 \to \mathbb{Z} \to \mathcal{O}_M \to \mathcal{O}_M^\times \to 0$.

$$H^1(M;\mathcal{O}_M) \longrightarrow H^1(M;\mathcal{O}_M^\times) \longrightarrow H^2(M;\mathbb{Z}) \longrightarrow H^2(M;\mathcal{O}_M)$$

$$H^1(M;\mathcal{O}_M^\times) \twoheadrightarrow H^2(M;\mathbb{Z}) \text{ surjective.} \qquad H^2(M;\mathcal{O}_M) \cong \boxed{H^{0,2}(M;\mathbb{C})} = 0$$

So, pick a class $[\omega] \in H^2(M;\mathbb{Z})$ in the Kähler cone.

$\Rightarrow$ $^\exists \mathcal{L}_\omega$ on $M$. so $c_1(\mathcal{L}_\omega) = [\omega]$.

Use $\Gamma(M;\mathcal{L}_\omega)$ to construct $M \longrightarrow \mathbb{P}^{N = \dim_{\mathbb{C}}[\Gamma(M;\mathcal{L}_\omega)]-1}$.

key words :
- Kodaira embedding theorem (above)
- Chow's theorem
  ↳ [every complex analytic submanifold of a projective space is algebraic.]

(my alg-geom lecture note §4.3)

(Refs. for p.14, p.15:
Polchinski Chaudhuri Johnson th/9602052.
Giveon Kutasov th/9802067. (Chapt I + II))

## §1.4 D-brane

D-branes placed in a flat space-time.

Type IIA string theory

D0-brane. D2-brane. D4-brane. D6-brane. D8-brane
are stable.

$\eta \in \underline{\underline{16}}$ of $SO(9,1)$
$\eta' \in \underline{\underline{16}}'$ of $SO(9,1)$
} SUSY transformatn parameters of 10D (1,1)-SUSY SUGRA.
→ total : 32 SUSY charges.

Dp-brane stretched in
$x^0, x^1 \sim x^p$ directions
preserve SUSY charges with $\eta = \underbrace{\Gamma^{012\cdots p}}_{\text{odd }\Gamma\text{'s}}\eta'$ ——— ⊛
opposite chirality.

Type IIB string theory

Dp-branes with odd p's.

$\eta^1, \eta^2 \in \underline{\underline{16}}$ of $SO(9,1)$ SUSY Transformatn parameters.

A Dp-brane stretched in the $x^0, x^1 \sim x^p$ directions.
preserves SUSY charges with $\eta^1 = \underbrace{\Gamma^{012\cdots p}}_{\text{even }\Gamma\text{'s}}\eta^2$ ——— ⊛
the same chirality.

⊛ : can be derived from the algebra $\{Q^\dagger, Q\} = \Gamma^M P_M + \begin{pmatrix}\text{central}\\ \text{charges}\end{pmatrix}$
just like in Wess-Bagger Chap. I.
(must be written somewhere in Polchinski (Chap. 13))

☆ If a Dp-brane and a Dp'-brane coexist ...

the remaining SUSY changes are $\boxed{\begin{array}{l} \eta = \Gamma^{01\cdots p}\eta' \\ \& \ \ \eta = \Gamma^{01\cdots p'}\eta' \end{array}}$

$\boxed{\Rightarrow \ \ \Gamma^{(p'+1)(p'+2)\cdots p}\,\eta' = \eta' \text{ follows.}}$

Because $(\Gamma^1\Gamma^2)^2 = \Gamma^1\Gamma^2\Gamma^1\Gamma^2 = -(\Gamma^1)^2(\Gamma^2)^2 = -1$.

$(\Gamma^1\Gamma^2)$ has only two eg. values $\pm i$. $\Gamma^{12}\eta' \neq \eta'$.

$\Gamma^{1234}\ \Gamma^{1234} = (-)^{3+2+1}(\Gamma^1)^2(\Gamma^2)^2(\Gamma^3)^2(\Gamma^4)^2 = +1$.

$(\Gamma^{1234})$ has two eg. values $\pm 1$.

"So", the condition $\Gamma^{1234}\eta' = \eta'$

reduces the SUSY changes by half.

$\boxed{\text{D3} - \text{D7} \quad {}^{4D}\ N=2 \text{ SUSY gauge theory in } \mathbb{R}^{3,1}.}$

☆ If a Dp-brane and anothe Dp-brane intersect with an angle ...

the condition for remaining SUSY can be worked out

by taking the common subset of the preserved SUSY

changes of both D-branes.

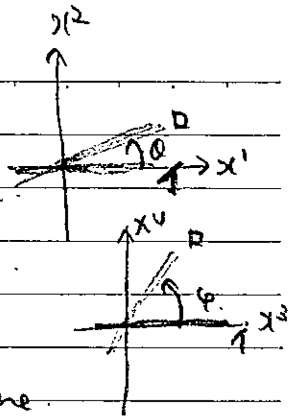
all of them δ-SUSY change system

| | | 0 | 1 ~ p | (p+1) | (p+2) | (p+3) | (p+4) | (p+5) ~ 9 |
|---|---|---|---|---|---|---|---|---|
| Dp − D(p+4) | Dp | 0 | 0 ---- 0 | x | x | x | x | x --- x |
| | D(p+4) | 0 | 0 ---- 0 | o | o | o | o | x ··· x |
| D(p+1) − D(p+3) | D(p+1) | 0 | 0 --- 0 | o | x | x | x | x --- x |
| | D(p+3) | 0 | 0 ---- 0 | x | o | o | o | x --- x |
| D(p+2) − D(p+2) | D(p+2) | 0 | 0 --- 0 | o | o | x | x | x --- x |
| | D(p+2) | 0 | 0 --- 0 | x | x | o | o | x -- x |

o: stretched (Neumann)

x: localized (Dirichlet)

T-duality.




## Generalization step ①

D(p+2)-brane ① stretched in the angle $\theta$ in the $x^1$-$x^2$ plane
angle $\varphi$ in the $x^3$-$x^4$ plane
in (p-2) more directions

D(p+2)-brane ② stretched in the angle $(-\theta)$ in the $x^1$-$x^2$ plane
angle $(-\varphi)$ in the $x^3$-$x^4$ plane
in the (p-2) directions common to ①.

$$\Rightarrow \begin{cases} \eta' = \Gamma^0 (\Gamma's)^{p-2} (\cos\varphi\,\Gamma^3 + \sin\varphi\,\Gamma^4)(\cos\theta\,\Gamma^1 + \sin\theta\,\Gamma^2)\,\eta \\ \eta' = \Gamma^0 (\Gamma's)^{p-2} (\cos\varphi\,\Gamma^3 - \sin\varphi\,\Gamma^4)(\cos\theta\,\Gamma^1 - \sin\theta\,\Gamma^2)\,\eta \end{cases}$$

$$\Rightarrow \boxed{\eta = (\cos\theta\,\Gamma^1 - \sin\theta\,\Gamma^2)(\cos\varphi\,\Gamma^3 - \sin\varphi\,\Gamma^4)(\cos\varphi\,\Gamma^3 + \sin\varphi\,\Gamma^4)(\cos\theta\,\Gamma^1 + \sin\theta\,\Gamma^2)\,\eta}$$

$$= (\cos\theta\,\Gamma^3 - \sin\varphi\,\Gamma^4)(\cos\varphi\,\Gamma^3 + \sin\varphi\,\Gamma^4)(\cos\theta\,\Gamma^1 - \sin\theta\,\Gamma^2)(\cos\theta\,\Gamma^1 + \sin\theta\,\Gamma^2)\,\eta$$

$$\Downarrow$$

use $\Gamma^{1,2} = 1 \otimes \tau^{1,2}$, $\Gamma^{3,4} = \tau^{1,2} \otimes \tau^3$

$$= \begin{pmatrix} e^{2i\varphi} & 0 \\ 0 & e^{-2i\varphi} \end{pmatrix} \otimes \begin{pmatrix} e^{2i\theta} & 0 \\ 0 & e^{-2i\theta} \end{pmatrix} \text{ on } \eta$$

$$\mathrm{diag}\left(e^{2i(\varphi+\theta)},\ e^{2i(\varphi-\theta)},\ e^{2i(-\varphi+\theta)},\ e^{-2i(\varphi+\theta)}\right) \text{ on } \eta.$$

$\Rightarrow$ when $\boxed{\theta = -\varphi}$ $\underline{SU(2)\text{ rotation}}$ on $\begin{pmatrix} x^1 + ix^2 \\ x^3 + ix^4 \end{pmatrix} \sim \mathbb{C}^2$

half of 16 SUSY charges remain unbroken

$\Rightarrow$ when $\boxed{\theta = \varphi}$ $\underline{SU(2)\text{ rotation}}$ on $\begin{pmatrix} x^1 + ix^3 \\ x^2 + ix^4 \end{pmatrix} \sim \mathbb{C}^2$

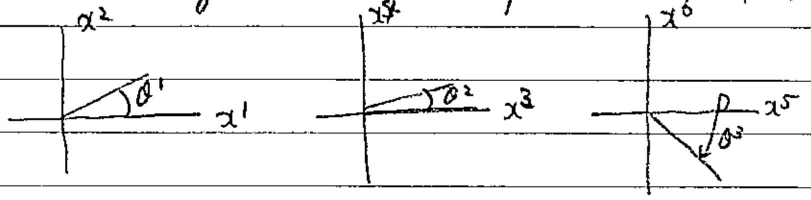
limit $(\pm 2\theta = \pm 2\varphi = \pm \pi/2) \Rightarrow$ D(p+2)-D(p+2) system in the prev. page.

Intersecting $\overset{N_\times}{\text{D(p+2)}}$-$\overset{M_\times}{\text{D(p+2)}}$ system : one hypermultiplet of 8-SUSY charges
$(N, \bar{M})$ repr. (+h.c.)
on $\mathbb{R}^{p,1}$ if one brane config is mapped to the other by an SU(2) rotation.

Direct computation of open string spectrum
$\Rightarrow$ degeneracy between the NS & R sectors in that case.

## Generalization step ②

similar argument and computation. D(p+3) − D(p+3) system.




$\theta^1+\theta^2+\theta^3=0.$

(SU(3) rotation)

$\Leftrightarrow \frac{1}{4} \times 16$SUSY unbroken.

(Berkooz Douglas Leigh th/9606139)

one chiral multiplet of 4-SUSY charges on $\mathbb{R}^{p,1}$.

## Generalization step Ⓐ

⊙ $W_6$ : $CY_3$ wrap
- D7-branes on one holomorphic surface ⇒ 4-SUSY
- D5-branes on one " curve ⇒ 4-SUSY
- D3-branes ⇒ 4-SUSY charges.

($\frac{1}{4}$ by SU(3) holonomy) × ($\frac{1}{2}$ by D-brane)

= $\frac{1}{8}$ × 32-SUSY charges.

⊙ $M_6$ : $CY_3$. wrap D6-branes on one slag 3-cycle ⇒ 4-SUSY.

Def. a 3-cycle $L \subset M_{CY_3}$ is Lagrangian if the symplectic form $\omega$ on $M$ becomes trivial when pulled back to $L$ and evaluated at each point in $L$.

It is further special Lagrangian if $\boxed{\mathrm{Re}(\Omega^{3,0})|_L = (\mathrm{const})(\mathrm{Im}(\Omega^{3,0})|_L) \text{ on } L}$

Refs.

Becker Becker Strominger th/9507158

Ooguri Oz Yin th/9606112

Hori Iqbal Vafa th/0005247

## Generalization Steps Ⓐ × ②

IIA/($CY_3 = M_6$) wrap $n_1$ D6-branes on slag $L_1$

and $n_2$ D6-branes on slag $L_2$

and ....

(need O6-planes too)

$\Rightarrow$ on $\mathbb{R}^{3,1}$ $\mathcal{N}=1$ SUSY w/ vector multiplets $U(n_1)\times U(n_2)\times\cdots$

$$\begin{cases} (L_1\cdot L_2) \text{ chiral multiplets in the } (n_1\otimes\bar{n}_2\otimes \mathbf{1}\cdots) \text{ repr.} \\ b_1(L_1) \quad " \quad " \quad \text{in the } (\text{adj.}\otimes \mathbf{1}, \otimes\mathbf{1}\cdots) \text{ repr.} \end{cases}$$

## Generalization Steps Ⓐ × ①

IIB/($CY_3 = W_6$) wrap $n_1$ D7-branes on a hol. surface $S_1$

and $n_2$ D7-branes on a hol. surface $S_2$

and ...

(need O7-planes too)

$\Rightarrow$ on $\mathbb{R}^{3,1}$ $\mathcal{N}=1$ SUSY w/ vector multiplets $U(n_1)\times U(n_2)\times\cdots$

$$\#[\text{chiral multiplets } (n_1\otimes\bar{n}_2)] - \#[\text{chiral } (\bar{n}_1\otimes n_2)]$$

$$= \int_{S_1\cdot S_2}\left(\left(\frac{``F"}{2\pi}\right)_{S_1} - \left(\frac{``F"}{2\pi}\right)_{S_2}\right)$$

intuition: hypermultiplet on $\mathbb{R}^{3,1}\times(S_1\cdot S_2)$.

the ground states of Landau levels = (magnetic flux on $(S_1\cdot S_2)$)

$\updownarrow$

massless states on $\mathbb{R}^{3,1}$

see references cited in

Watari Yanagida th/0402160.

side remark. Freed Witten th/9907189

adj. matter. Katz Sharpe th/0208104

Jockers Louis th/0502059

# §2. Introduction to String Duality

often in this context

multiple theoretical descriptions for one and the same vacuum.

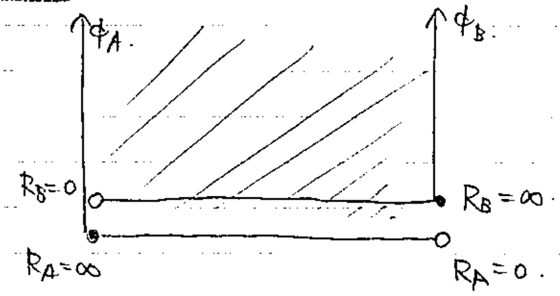
(rather than [(one description) $\xrightarrow{\text{RG flow}}$ (another description)

(one UV theory), (another UV theory) $\xrightarrow{\text{RG flow}}$ (common IR effective theory) ] )

(comprehensive view on particle pheno in string theory) (one description useful / easier than others.) (using the easier one to get math predict'n on the other)

## §2.1 T-duality

Example: $IIA/S^1 = IIB/S^1$. $\mathcal{M} = (\text{discrete})\backslash O(1,1;\mathbb{R})/O(1)\times O(1)$




common moduli space

the same spectrum

the same physics.

should not double count.

$$\left(R_A, \phi_A, \int_{S^1} C^{(1)}\right) \underset{\text{map}}{\rightleftarrows} \left(R_B, \phi_B, C^{(0)}\right)$$

Example: Het $E_8\times E_8$ — Het $SO(32)$ duality (Polchinski II. p. 78)

$$Het/S^1 : \left(g_{99}, \oint A, \phi\right)_{E_8E_8} \rightleftarrows \left(g_{99}, \oint A, \phi\right)_{SO(32)}$$

$$\boxed{\text{common moduli space} = (\text{discrete})\backslash O(1,17;\mathbb{R})/O(1)\times O(17)}$$

$(\times \mathbb{R}_{>0})$

local field redefinit'

cf. Theorem: Let $L$ be an even self-dual lattice of signature $(r_+, r_-)$.

Then $r_+ - r_- \equiv 0 \mod 8$. Moreover, if $r_+>0$ & $r_->0$, it is unique modulo isometry. So, $II_{1,1}\oplus(E_8\oplus E_8) \simeq II_{1,1}\oplus(D_{16};\mathbb{Z}_2)$ lattice isometry.

When $r_-=0$, $L=E_8$ if $r_+=8$, $L=\begin{cases}E_8\oplus E_8 \\ D_{16};\mathbb{Z}_2\end{cases}$ if $r_+=16$, there are 24 $L$'s if $r_+=24$.

No.

Date

<u>Example</u> ~~derivation using the field theory on world sheet $\Sigma$~~

Hori Vafa '00

A non-linear $\sigma$-model w/ the target space $M$:

When $\exists$ coordinate system $\pi: M \to B$

$S^1$ (or $T^n$) ($\rho$'s)

metric on $M$. (fiber. $\varphi \sim \varphi + 2\pi$)

$$\boxed{ds_M^2 = f(\rho)\,(d\varphi)^2 + ds_B^2}$$

$$\boxed{\mathcal{D}\rho\,\mathcal{D}\varphi \quad \text{and} \quad -S' = -\frac{1}{4\pi\alpha'}\int d^2\sigma \left(f(\rho)(\partial_\mu\varphi)(\partial^\mu\varphi) + \cdots\right)} \quad \mu = 0,1$$

$\Updownarrow$

$$\boxed{\mathcal{D}\rho\,\mathcal{D}J_\mu\,\mathcal{D}\theta \quad \text{and} \quad -S' = -\frac{1}{4\pi\alpha'}\int d^2\sigma \left(f(\rho) J_\mu J^\mu + \cdots\right) \overset{?}{\pm} \int d\theta \wedge J}$$

$\Downarrow$ integrate out $J$

$$\mathcal{D}\rho\,\mathcal{D}\theta \quad \text{and} \quad -S' = -\frac{1}{4\pi\alpha'}\int d^2\sigma \left(f(\rho)\left(J_\mu \pm^? \frac{4\pi\alpha'}{f(\rho)}(*d\theta)_\mu\right)\left(J^\mu \pm^? \frac{4\pi\alpha'}{f(\rho)}(*d\theta)^\mu\right) + \cdots\right)$$

$$- 4\pi\alpha' \int d^2\sigma \frac{1}{f(\rho)} (\partial_\mu\theta)(\partial^\mu\theta)$$

$\Downarrow$

$$\boxed{-S' = -4\pi\alpha' \int d^2\sigma \frac{1}{f(\rho)}(\partial_\mu\theta)(\partial^\mu\theta) - \frac{1}{4\pi\alpha'}\int d^2\sigma \,(\cdots)_{\rho's}}$$

just like in 4D converting electric $A_\mu$ to magnetic $\tilde{A}_\mu$

10D : $(p+1)$-form $C^{(p+1)}$ to $(7-p)$-form $\tilde{C}^{(7-p)}$

on spacetime.

- ~~not necessarily $M = (S^1 \text{ or } T^n) \times B$~~ fine!
- not just for spectrum but for OPE.

PLUS

§2.2

# M-theory and Type IIA String Theory

Refs. Polchinski Chap. 12 冒頭. Giveon Kutasov Chap. II.

M-theory : something whose L.E. effective theory is D=11 supergravity.

bosonic massless modes (d.o.f.) soliton

$(g_{mn}, C^{(3)}_{lmn})$. $\begin{cases} \text{M2-brane} \\ \text{M5-brane} \end{cases}$

- 11D SUGRA / $S^1$

$$\Longrightarrow \left[\begin{matrix} g_{\mu\nu} & g_{\mu,11} \to C^{(1)}_\mu & g_{11\,11} \to \phi \\ C^{(3)}_{\lambda\mu\nu} & C^{(3)}_{11\,\mu\nu} \to B_{\mu\nu} & \end{matrix}\right]$$

massless d.o.f. of Type IIA.

- M2-branes wrapped on $S^1 \longrightarrow$ F1
  - not 〃 〃 〃 $\longrightarrow$ D2
  - M5-branes wrapped on $S^1 \longrightarrow$ D4
  - not 〃 〃 〃 $\longrightarrow$ NS5

11D sugra.

$$S = \frac{1}{2\kappa_{11}^2}\int d^{11}x\sqrt{-g}\,R + \cdots \qquad \frac{1}{2\kappa_{11}^2} = \frac{(2\pi)}{\ell_{11}^9}$$

$$T_{M2} = \frac{2\pi}{\ell_{11}^3} \qquad T_{M5} = \frac{2\pi}{\ell_{11}^6}$$

IIA

Dp-brane

$$T_p = \frac{1}{(2\pi)^p(\alpha')^{\frac{p+1}{2}}}$$

$S^1$ compactification. radius $R_{11}$

- $S^1$-KK state mass $= \frac{1}{R_{11}}$ charged under $g_{\mu 11} = C^{(1)}_\mu$ D0
  $\hookrightarrow \frac{1}{\sqrt{\alpha'}\, g_{s,\mathrm{IIA}}} = T_0$.
- M2 on $S^1$

$$\frac{2\pi}{(\ell_{11})^3}\times(2\pi R_{11}) \longrightarrow \frac{1}{2\pi\alpha'}$$

dictionary between $(\ell_{11}, R_{11}) \longleftrightarrow (\alpha', g_{s,\mathrm{IIA}})$

then it follows that ....

M2 ~~on~~ w/o wrapping.

$$\frac{(2\pi)}{(l_{11})^3} \longrightarrow \frac{1}{2\pi\alpha'} \times \frac{1}{2\pi\sqrt{\alpha'}\, g_{s.IIA}} = \frac{1}{(2\pi)^2(\alpha')^{3/2} g_{s.IIA}} = T_{D2}$$

M5 wrapping

$$\cancel{\frac{2\pi}{(l_{11})^6} \rightarrow \frac{1}{(2\pi)} \times}$$

$$\frac{2\pi}{(l_{11})^6} \times (2\pi R_{11}) \longrightarrow \frac{T_{D2} \times T_{F1}}{(2\pi)} = \frac{1}{(2\pi)^4(\alpha')^{5/2} g_{s.IIA}} = T_{D}$$

M5 w/o wrapping

$$\frac{2\pi}{(l_{11})^6} \longrightarrow \frac{(T_{D2})^2}{2\pi} = \frac{1}{(2\pi)^5(\alpha')^3 g_{s.IIA}^2}$$

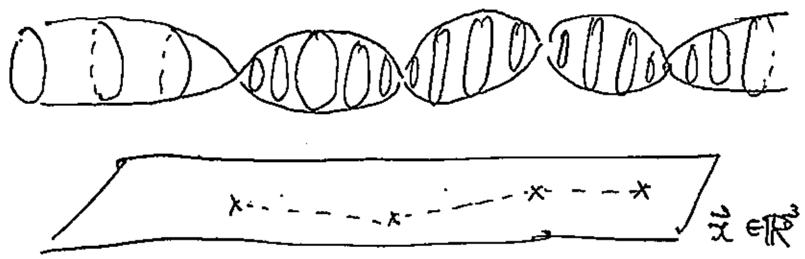
★ multi centered Taub-NUT space. for M-theory — D6-brane in IIA

$\left(\vec{x}\in\mathbb{R}^3,\ \tau\in[0,\underset{\times m}{4\pi}]\right)$ coordinates. (Sen. th/9707123.)

$$ds^2 = U(\vec{x})^{-1}\left(d\tau + A_i(\vec{x})dx^i\right)^2 + U(\vec{x})\, dx^i dx^i$$

$$\begin{cases} U(\vec{x}) = 1 + \sum_{I=1}^{N} \dfrac{m}{|\vec{x}-\vec{x}_I|} \\ \vec{\nabla}\times\vec{A} = -\vec{\nabla} U(\vec{x}) \end{cases}$$

smooth if $\vec{x}_I$'s $(I=1\sim N)$ are distinct from one another




- large $|\vec{x}|$ :
  $U(\vec{x})\simeq 1$.
  const. radius.
  $M \longrightarrow$ IIA.
- $\vec{x}\sim\vec{x}_I$ :
  $U \longrightarrow +\infty$
  $U^{-1}\longrightarrow 0$
  $S^1$ radius $\Rightarrow 0$.

$S^1$-fibration over intervals between $\{\vec{x}_I\}$'s

N-center $\Rightarrow$ (N-1)-topological 2-cycles.

M-theory ($g_{mn}$, $C^{(3)}_{lmn}$) on $A_{N-1}$-type (N-center) Taub-NUT space.

~~$\Rightarrow$ ...~~

M2-brane on the 2-cycles.

$\Rightarrow$ openstring (F1 string)

end points (center $\vec{x}_I \in \mathbb{R}^3$). D6-branes.

SU(N) gauge theory ($A_{N-1}$)

★ If we take $U(\vec{x}) = \sum_{i=1}^{N}\dfrac{m}{|\vec{x}-\vec{x}_I|}$ instead.

That's $\mathbb{C}^2/\mathbb{Z}_N$ in the limit of $\vec{x}_I\to\vec{0}$.

intersection form

$$\begin{bmatrix} -2 & 1 & & & 0 \\ 1 & -2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ 0 & & & 1 & -2 \end{bmatrix} \Rightarrow (-1)\times \text{ Cartan matrix of } A_{N-1}$$

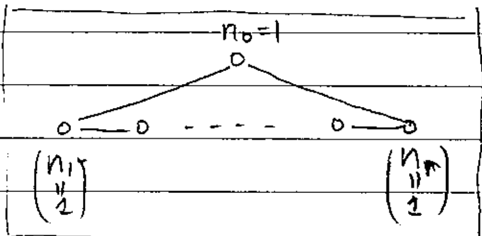
~~$N\times$ D6 in IIA $\simeq$ $A_{N-1}$ singularity in 11D SUGRA~~

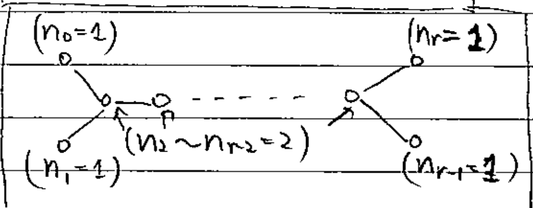
~~O6-plane in IIA $\Longleftrightarrow$ Atiyah-Hitchin metric~~
~~instead of the Taub-NUT metric.~~
see Sen's paper for the expression.

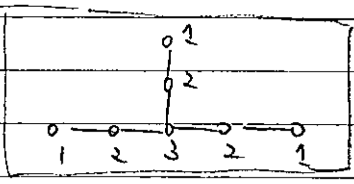
(memo) (almost locally Euclidean)

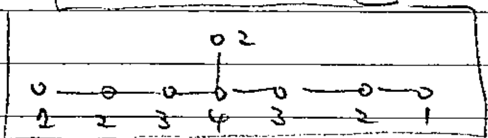
~~The ALE space construction via hyperkähler quotient~~

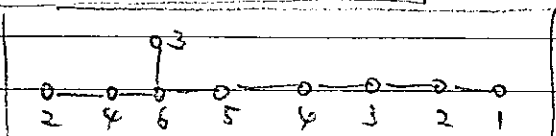
[Kronheimer J. Diff. Geom. $\underline{29}$ ('89) 665] ~~(as the moduli space of 8-SUSY quiver gauge theories)~~




$\boxed{\sum_{i=1}^{r} n_i \alpha_i = (\text{maximal root})}$

nodes $(i)$ $\Longleftrightarrow$ $U(n_i)$ vector multiplet.
Fayet-Iliopoulos parameter ($\vec{\xi}_i \in \mathbb{R}^3$)

edge $\langle ij\rangle$ $\Longleftrightarrow$ matter hyper multiplet in the bi-fund repr.

$\left(\sum_{i=0}^{r} n_i \vec{\xi}_i = 0\right) \Rightarrow 3r$ real parameters

dimension counting in $\underline{\mathbb{C}}$.

$A_r$ : [(matter) $2\times(r+1)$] $-$ [(gauge) $2\times r$] $= \underline{\underline{2\text{cpx}}}$

$D_r$ : [(matter) $4\times 4+8\times(r-4)$] $-$ [(gauge) $2\times\begin{pmatrix}3\\+\\4\times(r-3)\end{pmatrix}$] $= \underline{\underline{2\text{cpx}}}$.

$E_6$ : [(matter) $4\times 3 + 12\times 3$] $-$ [(gauge) $2\times(2+4\times3+9\times1)$]
$= 48-46 = \underline{\underline{2\text{cpx}}}$