

§2.2. M-theory and Type IIA String Theory

Refs.

Polchinski §12.
(beginning)Giveon Kutasov.
Chap. II.

M-theory: something whose L.T. eff. theory
is D=11 supergravity.

bosonic massless D.O.F. + solitons

$$(g_{mn}, C_{lmn}^{(3)})$$

$$M2\text{-brane } (C^{(3)}) + M5\text{-brane } (C^{(6)})$$

$$dC^{(3)} \leftrightarrow *dC^{(6)}$$

$$(M\text{-theory})$$

$$11D\text{ SUGRA} / \mathbb{Z}_2 = \text{Type IIA}$$

$$\left[\begin{array}{ccc} g_{\mu\nu} & g_{\mu,10} \Rightarrow C_\mu^{(1)} & g_{1010} \Rightarrow \phi_{0A} \\ & C_{\lambda\mu\nu}^{(3)} & C_{10,\mu\nu}^{(3)} \Rightarrow B_{\mu\nu} \end{array} \right]$$

massless D.O.F. of
Type IIA SUGRA.

$$\text{moduli (vev)} \quad [\langle g_{1010} \rangle \Leftrightarrow R_{10} + l_{11}] \text{ vs } [\langle e^{\phi_{0A}} \rangle = g_{s,A} + l_s]$$

$$\text{The dictionary: } (l_{11})^3 = (l_s)^3 g_{s,A}$$

$$(2\pi R_{10}) = l_s g_{s,A}$$

11D SUGRA

$$S > \frac{2\pi}{l_{11}^9} \int d^{11}x \sqrt{-g} R + \dots$$

$$T_{M2} = \frac{2\pi}{(l_{11})^3}$$

$$T_{M5} = \frac{2\pi}{(l_{11})^6}$$

Type IIA

$$S > \frac{2\pi}{l_s^8 g_{s,A}^2} \int d^{10}x \sqrt{-g} e^{-2\phi} R + \dots$$

$$T_{Dp} = \frac{2\pi}{(l_s)^{p+1} g_{s,A}}$$

$$T_{F1} = \frac{2\pi}{(l_s)^2}$$

10D Newton const.

$$\frac{(2\pi R_{10})}{l_{11}^9} = \frac{1}{(l_s)^4 g_{s,A}^2}$$

$$T_{NS5} = \frac{2\pi}{(l_s)^6 g_{s,A}^2}$$

→ KK-mass.

$$\frac{1}{R_{10}} = \frac{2\pi}{l_s g_{s,A}} = T_{D0}$$

$g_{\mu 10} = C_\mu^{(1)}$
changed.

M2 wrapped

$$\frac{2\pi \frac{2\pi R_{10}}{(l_{11})^3}}{(l_{11})^3} = 2\pi \frac{1}{(l_s)^2} = T_{F1}$$

M2 not wrapped

$$2\pi \frac{1}{(l_{11})^3} = 2\pi \frac{1}{(l_s)^3 g_{s,A}} = T_{D2}$$

M5 wrapped

$$2\pi \frac{2\pi R_{10}}{(l_{11})^6} = 2\pi \frac{1}{(l_s)^4 g_{s,A}} = T_{D4}$$

M5 not wrapped

$$2\pi \frac{1}{(l_{11})^6} = 2\pi \frac{1}{(l_s)^6 g_{s,A}^2} = T_{NS5}$$

* multi centered Taub-NUT space. for M-theory — D6-brane in $\mathbb{I}A$

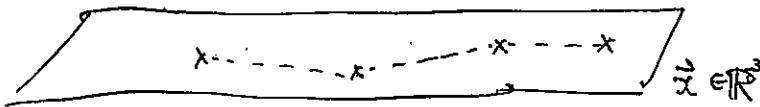
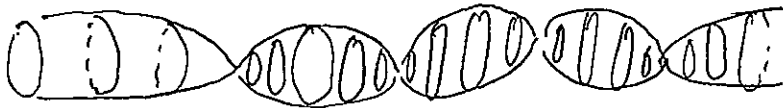
$(\vec{x} \in \mathbb{R}^3, \tau \in [0, 4\pi])$ coordinates.

(Sen. th/9707123.)

$$ds^2 = U(\vec{x})^{-1} (d\tau + A_i(\vec{x}) dx^i)^2 + U(\vec{x}) dx^i dx^i$$

$$\begin{cases} U(\vec{x}) = 1 + \sum_{I=1}^N \frac{m_I}{|\vec{x} - \vec{x}_I|} \\ \vec{\nabla} \times \vec{A} = - \vec{\nabla} U(\vec{x}) \end{cases}$$

smooth if \vec{x}_I 's ($I=1 \sim N$) are distinct from one another



S^1 -fibration over intervals between $\{\vec{x}_I\}$'s

N -center $\Rightarrow (N-1)$ -topological 2-cycles.

• large $|\vec{x}|$:

$$U(\vec{x}) \approx 1.$$

const. radius.

$$M \rightarrow \mathbb{I}A.$$

• $\vec{x} \sim \vec{x}_I$:

$$U \rightarrow +\infty$$

$$U^{-1} \rightarrow 0$$

$$S^1 \text{ radius} \Rightarrow 0.$$

M-theory $(g_{mn}, C_{lmn}^{(3)})$ on A_{N-1} -type. (N -center) Taub-NUT space.

$$\Rightarrow \sqrt{\frac{2}{3}} \sum_{I=1}^{N-1} \frac{1}{|\vec{x} - \vec{x}_I|} \quad \vec{x}_I \in \mathbb{R}^3$$

M2-brane on the 2-cycles.

\Rightarrow openstring (F1 string)

end points (center $\vec{x}_I \in \mathbb{R}^3$) . D6-branes.

$S^1 U(N)$ gauge theory (A_{N-1})

* If we take $U(\vec{x}) = \sum_{I=1}^N \frac{m_I}{|\vec{x} - \vec{x}_I|}$ instead.

$$2m = (\log 5)^{DA}$$

That's $\mathbb{C}^2 / \mathbb{Z}_N$ in the limit of $\vec{x}_I \rightarrow 0$.

Intersection form

$$\begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & & & \\ & & \ddots & & \\ 0 & & & -2 & 1 \\ & & & 1 & -2 \end{bmatrix}$$

$\Rightarrow (-1) \times$ Cartan matrix of A_{N-1} .

$N \times D6$ in IIA \Leftrightarrow (multicentered Taub-NUT in M_2/S^1) \rightarrow ALE space type A_{N-1}
(or E/\mathbb{Z}_N singularity) in IID SUGRA

Ob-plane in IIA \Leftrightarrow Atiyah-Hitchin metric

\downarrow instead of the Taub-NUT metric.

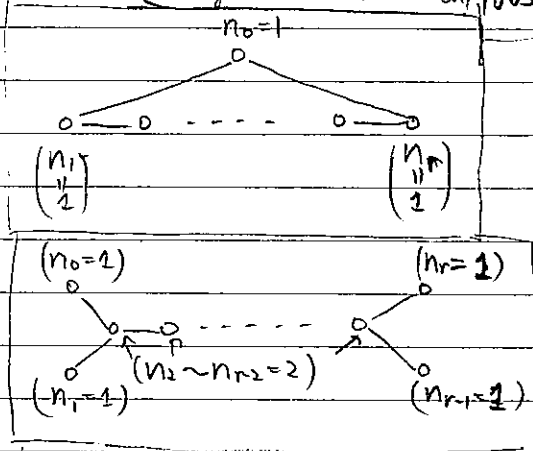
see Sen's paper for the expression.

(memo) \rightarrow (asymptotically locally Euclidean)

The ALE space construction via hyperkähler quotient

[Kronheimer J. Diff. Geom. 29 (89)]
[Douglas Moore ch/9603167 665]

(as the moduli space of $\mathcal{N}=2$ SUSY quiver gauge theories)



nodes $\Leftrightarrow U(n_i)$ vector multiplet

Fayet-Iliopoulos parameter
($\vec{E}_i \in \mathbb{R}^3$)

edges \Leftrightarrow matter hyper multiplet
in the bi-fund repr.

($\sum_{i=0}^r n_i \vec{E}_i = 0$) \Rightarrow $3r$ real parameters

dimension counting in \mathbb{C} .

$$A_r: [\text{matter}] \ 2 \times (r+1) - [\text{gauge}] \ 2 \times r = 2 \text{ cpx}$$

$$D_r: [\text{matter}] \ 4 \times 4 + 8 \times (r-4) - [\text{gauge}] \ 2 \times \left(\frac{3}{2} + 4 \times (r-3) \right)$$

$$E_6: [\text{matter}] \ 4 \times 3 + 12 \times 3 - [\text{gauge}] \ 2 \times (2 + 4 \times 3 + 9 \times 1) = 2 \text{ cpx}$$

$$= 48 - 46 = 2 \text{ cpx}$$

$$\sum_{i=1}^r n_i \alpha_i = (\text{maximal root})$$

The singular limit of the ALE space of ADE-type

of A_{N-1} -type = The metric $ds^2 = dz_1 \otimes d\bar{z}_1 + dz_2 \otimes d\bar{z}_2$ on $\mathbb{C}^2 = \{z_1, z_2\}$
quotient by $z_N \in SU(2)$
 $(z_1, z_2) \mapsto (z_N z_1, z_N^{-1} z_2)$

\Rightarrow descript'n in alg geometry (forget the Kähler metric)
and focus on cpx str.

z_N -invariants

$$\left[\mathbb{C}[(z_1)^N, (z_2)^N, (z_1 z_2)] / ((z_1)^N \cdot (z_2)^N - (z_1 z_2)^N) \right]$$

$$\cong \mathbb{C}[x, y, z] / (xy - z^N)$$

homogeneous
degree

$$\frac{N/2 \quad N/2 \quad 1}{N+1} \quad N$$

D_N singularity $\mathbb{C}[x, y, z] / (-y^2 + x^2 z + z^{N-1})$

$$\frac{N-2, N-1, 2}{2N-1} \quad 2N-2$$

E_6 $\mathbb{C}[x, y, z] / (-y^2 + x^3 + z^4)$

$$\frac{4: 6: 3}{13} \quad 12$$

E_7 $\mathbb{C}[x, y, z] / (-y^2 + x^3 + xz^3)$

$$\frac{6: 9: 4}{19} \quad 18$$

E_8 $\mathbb{C}[x, y, z] / (-y^2 + x^3 + z^5)$

$$\frac{10: 15: 6}{31} \quad 30$$

① $\exists \Gamma \subset SU(2)$ \mathbb{C}^2/Γ for D_N, E_6, E_7, E_8 .

(朝倉書店 "特異点と加群"
松澤 淳一)

② $\sum_{z_j z} (\text{homog degree}) = (\text{homog. deg of hyper-surf. eq}) + 1$. \rightarrow $h_{\text{dual}}^1 = \text{rank}$
(the dual coxeter #)

③ known as "Kleinian singularity" "du Val singularity" "rational double point singularity".

(keywords to Google for more information)

§2.3 F-theory and Type IIB String Theory

Three observations

$$\textcircled{1} \quad \boxed{M\text{-theory}/T^2 = IIB/\alpha'^1} \Leftrightarrow \left[(M\text{-theory}/\alpha'^2) = IIA \right] \times \left[IIA/\alpha'^1 = IIB/\alpha'^1 \right]_{(T\text{-dual})}$$

3 $SO(8,1)$ -preserving moduli parameters

$$\left(T^2_{\text{cpx str}} + T^2_{\text{vol}} \right)_{\mathbb{R}^n} \quad \text{vs} \quad \left((C^{(0)} + i e^{-\Phi_0}) + R_B \right)_{\mathbb{R}^5}$$

dictionary in a 2-dim. subspace

$T^2 \text{ is rectangular} \quad \longleftrightarrow \quad \langle C^{(0)} \rangle = 0$	$T\text{-duality reln's}$
$\left\{ \begin{array}{l} \frac{(L_{11})^3}{(2\pi R_9)(2\pi R_{10})} = (2\pi R_B) \\ \left(0 + i \frac{R_9}{R_{10}} \right) = \left(0 + i \frac{1}{g_{s,B}} \right) \end{array} \right\}$	$\text{use } R_9 R_B = l_s^2 / (2\pi)^2$
$\left(\langle g_{10,9} \rangle \Leftrightarrow \langle C^{(1)}_{10,9} \rangle \Leftrightarrow \langle C^{(0)} \rangle \right)$	$\left\{ \begin{array}{l} \frac{2\pi R_9}{g_{s,A}} = \frac{2\pi R_B}{g_{s,B}} \\ (R_s: \text{common to IIA \& IIB}) \end{array} \right\}$
	to derive.

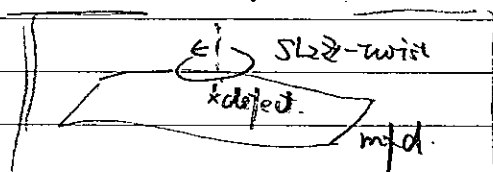
SO... the small (T^2 -vol.) limit of 11D SUGRA

becomes the 10D IIB SUGRA ($R_B \gg \infty$) w/ finite $\langle C^{(0)} + i e^{-\Phi} \rangle$

$\textcircled{2}$ IIB SUGRA: has $SL_2 \times$ symmetry

$$(F_1, D_1) \begin{pmatrix} B^{(2)} \\ C^{(1,2)} \end{pmatrix} \quad (NS5, D5) \begin{pmatrix} \tilde{B}^{(6)} \\ \tilde{C}^{(6)} \end{pmatrix} \quad (C^{(0)} + i e^{-\Phi}) \text{ kinetic term in 10D IIB SUGRA act'n. (Einstein frame)}$$

think of a compactification w/ an $SL_2 \times$ -twist?



$\textcircled{3}$ Bianchi identity $\boxed{dG^{(p+2)} = l_s^{-(1+p)} \delta^{(3+p)}}$ (D(6-p)-brane is a magnetic charge of the RR (p+1)-form field)

for a D7-brane ($p=-1$) $\Delta C^{(0)} = \oint_{\text{circle}} G^{(1)} = \int_{\text{area}} dG^{(1)} = \int \delta^{(1)} = 1$ not periodic around a D7.

$\textcircled{4}$ The Einstein frame. IIB equation of motion

$$\boxed{2\bar{\partial} \tau + \frac{2(\partial \tau)(\bar{\partial} \tau)}{(\bar{\tau} - \tau)} = 0} \Rightarrow \text{holomorphic } (C^{(0)} + i e^{-\Phi}) \text{ on } W_{n-1} \text{ or } B_{n-1}$$

$$(\tau := C^{(0)} + i e^{-\Phi}) \quad (\text{Nucl. Phys. B337 (1990) ? Greene et al.})$$

F-theory / (elliptic fibered CY_n) \supset Type IIB / (CY_{n-1} / \mathbb{Z}_2 (orientifold))
 \parallel
 W

$$\pi: Y \rightarrow B_{n-1}$$

ell. fibr.

$$R: W \rightarrow W \text{ inv.}$$

$$R^2 = \text{id}_W.$$

$$\left(\pi: Y \rightarrow (B \approx W / \langle 1, R \rangle) \right)$$

elliptic fibrat'n.

The elliptic fiber keeps track of the $SL_2 \mathbb{Z}$ -twist over B_{n-1} .

D7-branes @ $\{f(z)=0\} \subset B_{n-1}$

$$y^2 = x^2 - f(z) + x^3 \quad (z \in \mathbb{C} \subset B_{n-1})$$

$$\left(\text{when } f(z) = z^N \quad y^2 = x^2 - z^N: A_{N-1} \text{ singularity} \right)$$

$\Leftrightarrow N \times \text{D7-branes}$

one D7-plane + 8 (+4) D7-branes @ $z=0$

$$y^2 = x^3 + x z^2 + z^3 \quad (z \in \mathbb{C} \subset B_{n-1})$$

$$\Leftrightarrow \left(\begin{array}{l} y^2 = (x')^3 + (x')^2 z + z^3 \quad D_4 \text{ singularity} \\ (x = x' + \frac{z}{3}) \end{array} \right) \downarrow$$

$SO(8)$

$$\left(\begin{array}{l} T^2: \eta^2 = \xi^3 + f\xi + g \\ W \supset \mathbb{C} \ni u \end{array} \right] \mathbb{Z}_2 \langle \mathbb{Z}R \rangle \quad \begin{array}{l} R: u \rightarrow -u \\ \mathbb{Z}: \eta \rightarrow -\eta \end{array}$$

$$\Rightarrow \underbrace{\eta = (\eta u^3) \quad x = (\xi u^2) \quad z = u^2 \in \mathbb{C} \subset B}_{\text{inv. under } (\mathbb{Z}R)}$$

$$| y^2 = x^3 + f x z^2 + g z^3 |$$

$$\left[\frac{T^2_{(\mathbb{C}^2 + i\mathbb{R})} \times (T^2 = W)}{\mathbb{Z}_2 \langle \mathbb{Z}R \rangle} \right] = K3 \quad (Kum(T^2 \times T^2))$$

(Sen 2h/9605150)

§2.4 Het-II duality

Het/ T^4 - IIA/ $K3$ duality (16-SUSY) $K3: CY_2$

Narain compactification.

$$\left(\frac{O(4) \times O(20)}{O(4,20)} / \text{discrete} \right) \text{ for Het}$$

On the other hand, in IIA

moduli space of metric on $K3$

$$\left(\frac{SO(3) \times SO(19)}{SO(3,19)} / \text{discrete} \right)$$

Aspinwall th/9611337

$$\left\{ \begin{aligned} h^0(K3) &= h^4(K3) = 1 \\ h^2(K3) &= 22 \end{aligned} \right.$$

- B-field on the 22 2-cycles.
- 1 combination from ϕ_{10D} & $vol(K3)$.

$$\left(\frac{SO(4) \times SO(20)}{SO(4,20)} / \text{discr.} \right) \text{ for IIA}$$

$$R^m \rightarrow dx^k \wedge dy^l$$

$SU(2)$ -valued

not $SO(4)$ -valued.

$$\langle dy^k \omega_k^{\alpha\beta} \rangle$$

half non-trivial.

$CY_3: SU(3)$ -valued.

not $SO(6)$ -valued

$$SO(6)\text{-spinor} = SU(4)\text{-}\frac{1}{2}$$

$$\downarrow SU(3)$$

$$\left(\langle \omega \rangle \right) \frac{3}{4} \text{ non-trivial}$$

$\frac{1}{4}$ unbroken.

dimensional reduction. (massless spectrum in 6D eff-theory)

Het side.

$$\begin{aligned} g_{\mu\nu} & \quad g_{\mu i} \\ B_{\mu\nu} & \quad B_{\mu i} \\ \phi & \end{aligned}$$

$$16 \times A_\mu$$

$$\downarrow 24 \text{ vector}$$

$$\begin{aligned} & 4 \times 4 \text{ scalars } (B_{ij}, g_{ij}) \\ & 16 \times 4 \text{ scalar (Wilson line)} \end{aligned}$$

= 80 scalars.

IIA side.

$$\begin{aligned} g_{\mu\nu} \\ B_{\mu\nu} \\ \phi' \end{aligned}$$

$$\begin{aligned} & C^{(3)} \text{ on } 22 \text{ 2-cycles} \\ & \downarrow \\ & 22 \times \text{vector} \end{aligned}$$

$$\frac{SO(4,20)}{SO(4) \times SO(20)} \text{ scalars}$$

$$C_\mu^{(1)}$$

$$\downarrow 24 \text{ vectors}$$

dual of $C_{\mu\nu}^{(3)}$ 1-form

[the same massless spectrum in 6D.]

unbroken (enhanced) symmetry in Het.

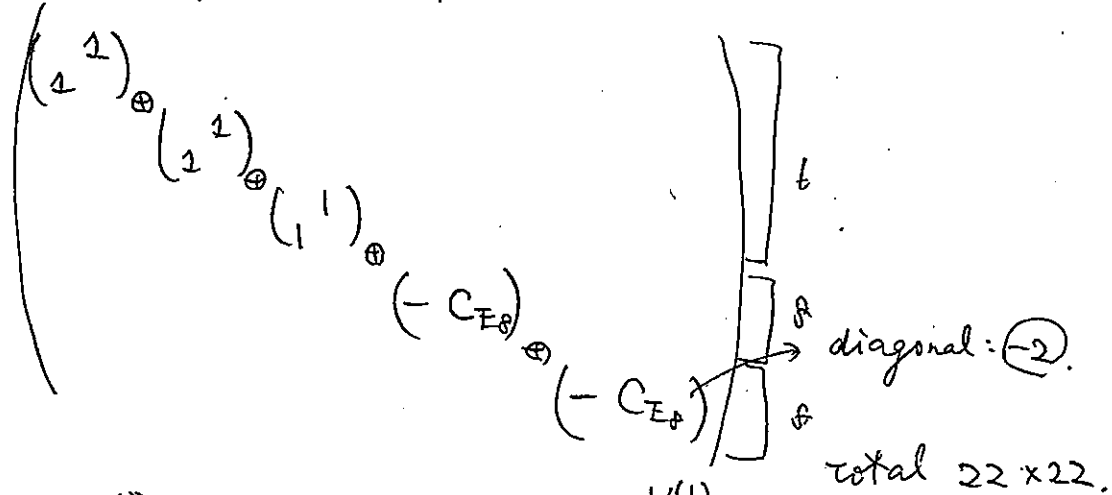
there must also be in IIA ?

(in high codim. subspace of $O(4,20)$ moduli space,
 $O(4) \times O(20)$)



(for a special choice of moduli of K3 (W, B))

Intersection form of 2-cycles. in K3.



$C^{(3)}$ on these 2-cycles \Rightarrow $U(1)$ vector in D=6 eff. theory.

D2-branes wrapped on these 2-cycles \Rightarrow charged

\Rightarrow W-boson's of non-Abelian gauge theory under these $U(1)$'s.

from SUSY 16-SUSY vector multiplet

superYM (Witten ch/9603150)

non-Abelian enhanced

when those cycles collapsed to a point (0-size)

- A-D-E singularity (isolated singul. pts in cpx surface)
- ALF space of A-D-E type.
- (multi centered Taub-NUT space of A-D-E type.)

Intersection form

$$\begin{pmatrix} -2 & 1 & & & 0 \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & 1 & -2 & 1 \\ 0 & & & 1 & -2 \end{pmatrix} \Rightarrow (-1) \times \text{Cartan matrix of } A_{N-1}.$$

Atiyah-Hitchin metric \Rightarrow D6-brane + O6-plane.

○ ~~Het~~ Het/ T^3 — M/ K^3 duality by "rolling up" (Acharya Witten th/0109152)

in 7D eff. theory

$$10-3 = 7D \text{ vs } 11-4 = 7D. \\ 16 \text{ SUSY} \text{ vs } 32 \times \frac{1}{2} = 16 \text{ SUSY}.$$

metric 7D. g_{mn}

2-form 7D. B^{Het} or C_{7D}^M

vector 7D. $\begin{pmatrix} 16 \text{ Cartan} \\ + \\ \text{KK vector} \\ 3(T^2) \times 2(g_{mn} \& B_{mn}) \end{pmatrix}$ or $\begin{pmatrix} C_{mab}^{(3)} \rightarrow 2\text{-cycles in } K^3 \\ h^2(K^3) = 22 \end{pmatrix}$

○ scalar 7D. $\begin{pmatrix} g_{\text{Het}}^{7D} \\ + \\ 3 \times 19 \text{ Narain moduli} \\ \approx 3 \times 16 \text{ Wilson line} \\ + 3 \times 3 \text{ } g_{ab}, B_{ab} \end{pmatrix}$ or $\begin{pmatrix} \text{metric on } K^3 \\ SO(3,19) / SO(3) \times SO(19) \\ + \text{ over all volume.} \end{pmatrix}$

unbroken $U(1)^{22}$. @ generic pt in the moduli space.

enhanced sym. @ special pts

$\begin{cases} \text{Het: vanishing Wilson lines} \\ M: \text{collapsed 2-cycles.} \end{cases}$

also $\text{Het}/T^2 = F/\ell_K^2$ by rolling up.

* M-theory/ T^2 $=$ IIB/ S^1 : verify that $T^2 \xrightarrow{2\pi R_B} M \xrightarrow{2\pi R_A} \Rightarrow \frac{1}{g_s} = \frac{R_B}{R_A}$ in IIB.

§2.5 Introduction to Mirror Symmetry

$$\mathbb{I}A/(C\Upsilon_3=M)$$

$$\mathbb{I}B/(C\Upsilon_3=W)$$

$h^{1,1}(M)$ vect. multiplets

$h^{2,1}(M)+1$ hypermultiplets

$h^{2,1}(W)$ vect. multiplets

$(h^{1,1}(W)+1)$ hypermultiplets

DA dilation

fluctuation field

DB dilation

fluctuation field

4D $N=2$ SUSY field theory

[2 derivatives / 1 derivative, 2 fermions / 4 fermions]

(exact
@ $g=0$)

holomorphic "function" (called prepotential) \Rightarrow vect. multiplet
NLSM metric (quaternionic manifold \approx hyperkähler mfd)
4D $N=2$ SUGRA 9D $N=2$ rigid SUSY

mirror symmetry

for some class of M 's $\exists W$

$M \leftrightarrow W$ so that the 4D $N=2$ SUGRA are identical.

$$h^{1,1}(M) = h^{2,1}(W) \quad h^{2,1}(M) = h^{1,1}(W)$$

the prepotential on both sides agree.

when we set an appropriate map between

the $h^{1,1}(M)$ -dim mfd and the $h^{2,1}(W)$ -dim mfd.

(of the vector multiplet scalars)

the spectrum of the 4D $N=2$ SUSY BPS particles
on both sides agree.

Example

homogeneous fn of deg=5.

$$M := \left\{ [x^0:x^1:x^2:x^3:x^4] \in \mathbb{P}^4 \mid F^{(5)}(x^0, x^1, x^2, \dots, x^4) = 0 \right\} \leftarrow \begin{array}{l} \text{cpx str.} \\ \text{determined.} \end{array}$$

metric: $\mathbb{R}_{5,0}$ (the Kähler metric of \mathbb{P}^4 restricted on M)

(Fubini-Study metric)

$$h^{1,1}(M) = 1.$$

$$h^{2,1}(M) = 101 \quad (126 \text{ coefficients in } F^{(5)} - 25 \text{ GL}_5(\mathbb{C}) \text{ on } \mathbb{P}^4)$$

 $W :=$ the proper transform of

$$\left\{ [y^0:y^1:y^2:y^3:y^4] \in \mathbb{P}^4 \mid \left(\sum_{i=0}^4 a_i y_i^5 \right) + a_0 y_0 y_1 y_2 y_3 y_4 = 0 \right\} / (z_5 \times z_5 \times z_5)$$

$$h^{2,1}(W) = 1 \text{ by } \left(\frac{a_0 a_2 a_3 a_4 a_5}{a_0^5} \right)$$

$$\begin{array}{l} z_5 : [y^0:y^1:y^2:y^3:y^4] \mapsto [y^0:ws y^1:ws y^2:y^3:y^4] \\ z_5 : [y^0:y^1:y^2:y^3:y^4] \mapsto [y^0:ws y^1:y^2:ws y^3:y^4] \\ z_5 \mapsto [y^0:ws y^1:y^2:y^3:ws y^4] \end{array}$$

metric: $\mathbb{R}_{5,0} \times$ (the Kähler metric of \mathbb{P}^4)+ resolut'n parameters of the $z_5 \times z_5 \times z_5$ singularities

$$\Rightarrow h^{1,1}(W) = 101.$$

The vector multiplet moduli space ($h^{1,1}(M) = h^{1,1}(W) - 1 = \text{dime}$)

$$\text{coordinates} \begin{cases} \text{IIA: } \int_{\substack{\text{deg}=1 \\ \text{curve} \subset M}} (B+iJ) \frac{1}{(2\pi i)^2 \alpha'} =: t_{\text{IIA}} \\ \text{IIB: } z := \left(\frac{a_0 a_2 a_3 \dots a_5}{a_0^5} \right) \end{cases}$$

4D $N=2$ SUSY: $\exists \neq (t)$ so

$$\begin{pmatrix} x^0=1 & F_0=(2-t\partial_t)F \\ x^2=t & F_2=(\partial_t F) \end{pmatrix} \Rightarrow K = -\ln \left[i(x^2 \bar{F}_2 - \bar{x}^2 F_2) \right] \begin{array}{l} \text{vect. mult.} \\ \text{scalar kin.} \\ \text{term} \end{array}$$

IIA expectat'n

$$\boxed{t = t_{\text{IIA}}}$$

 $\begin{array}{l} \text{vect. mult.} \\ \text{vector gauge} \\ \text{coupling} \end{array}$

$$F = \frac{dabc}{3!} t^a t^b t^c - \frac{a_0 a_2 a_3 \dots a_5}{2} t^a t^b - \frac{ba}{25} t^a - \frac{\zeta(3)}{(2\pi i)^3} \frac{\chi(M)}{2} + \frac{K_B}{\beta_{30} (2\pi i)^3} e^{2\pi i \langle \beta, t \rangle}$$

(honest co-def
computation
possible for
individual β 's)dabc: intresed'n number on $H^2(M; \mathbb{Z})$ K_B : Gromov-Witten invariants for $\beta \in H_2(M; \mathbb{Z})$

PLUS

II B : 3-cycles in $H_3(W; \mathbb{Z})$ $\{\alpha_0, \alpha_1, \beta^0, \beta^2\}$ s.t. $\alpha_1 - \alpha_0 = 0$
 $\beta^2 \cdot \beta^0 = 0$
 $-\beta^0 \cdot \alpha_2 = \alpha_1 \cdot \beta^0 = \partial_1^T$

Ω_W : holomorphic (3,0)-form on $W \in H^{3,0}(W; \mathbb{C}) \cong H^0(W; \Lambda^3 T^*W)$

$$\left(\int \alpha_0 \Omega, \int \alpha_1 \Omega, \int \beta^0 \Omega, \int \beta^2 \Omega \right)^T \propto (x^0, x^1, F_0, F_2)^T$$

$$\Rightarrow \tau = \frac{\int \alpha_1 \Omega}{\int \alpha_0 \Omega}$$

$$\text{analogy: } T^2: (\mathbb{C}/\mathbb{Z}) / (\mathbb{Z} \oplus \tau \mathbb{Z})$$

\Rightarrow computation of the period integrals.

$$z \sim z+1 \sim z+\tau$$

$$\Omega_{z+\tau} = dz = \hat{\alpha} + \tau \hat{\beta}$$

$$\left(\int \alpha \Omega, \int \beta \Omega \right) = (1, \tau)$$

is τ -dep.

(τ - z dictionary)

prepotential \mathcal{F} as a fun of τ or z .

$\Omega^{(n,0)}$ for a hypersurface n -fold W ,

ambient space coordinates $x_i, i=1 \sim n+1$.

def. eg $f(x) = 0$

$$\Rightarrow \Omega^{(n,0)} \propto \text{Res}_{f=0} \left[\frac{dx_1 \wedge dx_2 \wedge \dots \wedge dx_{n+1}}{f(x)} \right]$$

eg. $n=1$. W as an elliptic curve $f(x,y) = 4x^3 - g_2x - g_3 - y^2$

$$\text{Res}_{f=0} \left[\frac{dx \wedge dy}{(4x^3 - g_2x - g_3 - y^2)} \right] = -\frac{1}{2} \frac{dy}{y} = -\frac{1}{2} \frac{(dy/dx)}{(dy/dx)} = -\frac{dx}{2}$$

$$\Omega_W = \text{Res} \left[\frac{dy_1 \wedge dy_2 \wedge dy_3 \wedge dy_4 \cdot a_0}{(a_1 y_1^5 + a_2 y_2^5 + a_3 y_3^5 + a_4 y_4^5 + a_5 + a_0 y_1 y_2 y_3 y_4)} \right]$$

$(\tau) : \mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z}) \rightarrow W$
 Weierstrass $p \sim f(\tau)$

$$\left[\partial_{a_1} \partial_{a_2} \partial_{a_3} \partial_{a_4} \partial_{a_5} - a_0 \partial_{a_0}^5 \frac{1}{a_0} \right] \Omega_W = 0.$$

$$\left(a_1 \partial_{a_1} = a_2 \partial_{a_2} = a_3 \partial_{a_3} = a_4 \partial_{a_4} = a_5 \partial_{a_5} = -\frac{1}{5} a_0 \partial_{a_0} \right) \text{ on any fun of } z = \frac{a_1 a_2 a_3 a_4 a_5}{a_0^5}$$

$$\Rightarrow \left[(z \partial_z)^5 - z(-5z \partial_z - 5)(-5z \partial_z - 4) - \dots - (-5z \partial_z - 1) \right] \int_{3\text{-cycle}} \Omega_W = 0.$$

$$(z \partial_z) \cdot \left[(z \partial_z)^4 + 5z(5z \partial_z + 4)(5z \partial_z + 3) - (5z \partial_z + 2) \right]$$

The four solutions to

$$(\zeta := -z)$$

$$\left[\partial^4 - 5\zeta(5\partial + 4)(5\partial + 3)(5\partial + 2)(5\partial + 1) \right] \pi(\zeta) = 0.$$

$$\cdot W_0 = 1 + 120\zeta + 113.400\zeta^2 + \dots$$

$$\cdot W_1 = \ln(\zeta) W_0(\zeta) + \tilde{W}_1; \quad \tilde{W}_1 = 770\zeta + 870.225\zeta^2 + \dots$$

$$\cdot W_2 = \frac{[\ln(\zeta)]^2}{2} W_0(\zeta) + \ln(\zeta) \tilde{W}_1(\zeta) + \tilde{W}_2(\zeta);$$

$$\tilde{W}_2(\zeta) = 575\zeta + \frac{8208175}{4}\zeta^2 + \dots$$

$$\cdot W_3 = \frac{[\ln(\zeta)]^3}{3!} W_0(\zeta) + \frac{[\ln(\zeta)]^2}{2} \tilde{W}_1(\zeta) + \ln(\zeta) \tilde{W}_2(\zeta) + \tilde{W}_3(\zeta)$$

$$\tilde{W}_3(\zeta) = -1150\zeta - \frac{3288375}{4}\zeta^2 + \dots$$

\Rightarrow period integrals.

$$\left(W_0, \frac{W_1}{2\pi i}, -\frac{5}{(2\pi i)^2} W_3 + \frac{200 \zeta(3)}{(2\pi i)^2} W_0 + \frac{50 \zeta(2)}{(2\pi i)^2} W_1, \frac{5}{(2\pi i)^2} W_2 + \frac{50 \zeta(2)}{(2\pi i)^2} W_0 \right)$$

$$= W_0 \times (1, t, (2-t\partial_t)F, \partial_t F)$$

$$\begin{cases} e^{2\pi i t} = \zeta + 770\zeta^2 + 1014275\zeta^3 + \dots \\ \zeta = e^{2\pi i t} - 770 e^{2 \cdot 2\pi i t} + 171525 e^{3 \cdot 2\pi i t} + \dots \end{cases}$$

$$F = \frac{5}{3!} t^3 - \frac{50}{24} t - \frac{\zeta(3)(-200)}{(2\pi i)^3} + \frac{1}{(2\pi i)^2} \left(2475 e^{2\pi i t} + \frac{8476875}{8} e^{2 \cdot 2\pi i t} + \dots \right)$$

The Gromov-Witten invariants can be read out by the IB calculation (by solving diff. eq.)

$$K_1 = 2475, \quad K_2 = \frac{8476875}{8}, \quad K_3 = \frac{8564575000}{27}$$

Gopakumar-Vafa invariants:
$$\begin{pmatrix} n_1^0 := K_1 = 2475 & n_2^0 := K_2 - \frac{K_1}{8} = 604350 & n_3^0 := K_3 - \frac{K_1}{27} = 317206375 \end{pmatrix}$$

§ 2.7

§ 2.2 (F)

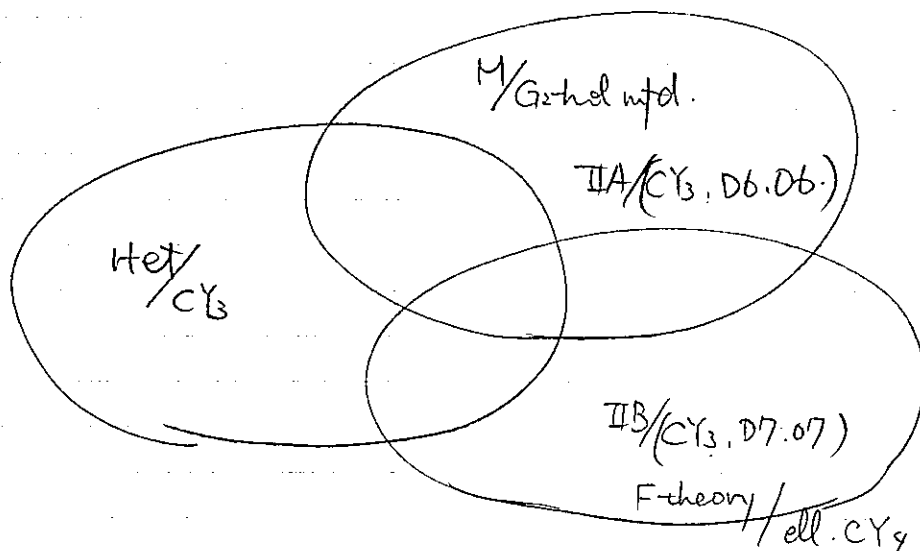
The adiabatic "argument"

The $\text{Het}/T^3 - \text{M-theory}/K_3$ duality suggests $\text{IIA}/(\text{CY}_3 \text{ w/ D6, 06})$

$\text{Het}/(\text{T}^3\text{-fibred } \text{CY}_3) - \text{M-theory}/(\text{K}_3\text{-fibred } G_2\text{-hol mfd})$ duality.

§ 2.2 (G)

any restriction on types of degenerate fibrat'n??



("The" landscape of string vacua with $D=4$ $N=1$ SUSY)

§ 2.6 BPS states of 4d space-time SUSY

4d $N=2$ SUSY algebra $I, J=1, 2$

$$\left\{ Q_{\alpha}^I, \bar{Q}_{\dot{\beta}, J} \right\} = 2 \sigma_{\alpha\dot{\alpha}}^{\mu} P_{\mu} \delta_{\dot{\beta}}^J \quad \left\{ Q_{\alpha}^I, Q_{\beta}^J \right\} = \epsilon_{\alpha\beta} \epsilon^{IJ} Z$$

central charge.

$$\langle P^0 \rangle \geq |Z| \quad (\text{BPS bound})$$

IIA / $(CY_3=M)$ \Rightarrow 4d $N=2$ SUSY, $U(1)^{h(M)+1}$: Dp-brane on p-cycle in M.

\Rightarrow U(1)-charged particle excitation. BPS

II B / $(CY_3=M)$ \nRightarrow D(p+3)-brane wrapped on p-cycle in M. \Rightarrow 4d $N=1$ SUSY gauge theory

$U(1)^{h(M)+1}$: orientifold projection \Rightarrow some of them odd

* Naive derivation of the formula for "Z"

IIA / $CY_3=M$ to 4d $N=2$

start from 10d IIA SUGRA SUSY algebra.

$$\begin{pmatrix} \{Q_-, Q_-^\dagger\} & \{Q_-, Q_+^\dagger\} \\ \{Q_+, Q_-^\dagger\} & \{Q_+, Q_+^\dagger\} \end{pmatrix} = \begin{bmatrix} (P^M P^0)_{UL} P_M & (*)_{UR} \\ (*)_{DL} & (P^M P^0)_{DR} P_M \end{bmatrix}$$

$$(*)_{UR} = (P^0 P^0)_{UR} Z^{(0)} + \frac{1}{2} (P^{MN} P^0)_{UR} Z_{MN}^{(2)} + \frac{1}{4!} (P^{KLMN} P^0 P^0)_{UR} Z_{KLMN}^{(4)}$$

$$(*)_{DL} = (P^0 P^0)_{DL} Z^{(0)} + \frac{1}{2} (P^{MN} P^0)_{DL} Z_{MN}^{(2)} + \frac{1}{4!} (P^{KLMN} P^0 P^0)_{DL} Z_{KLMN}^{(4)}$$

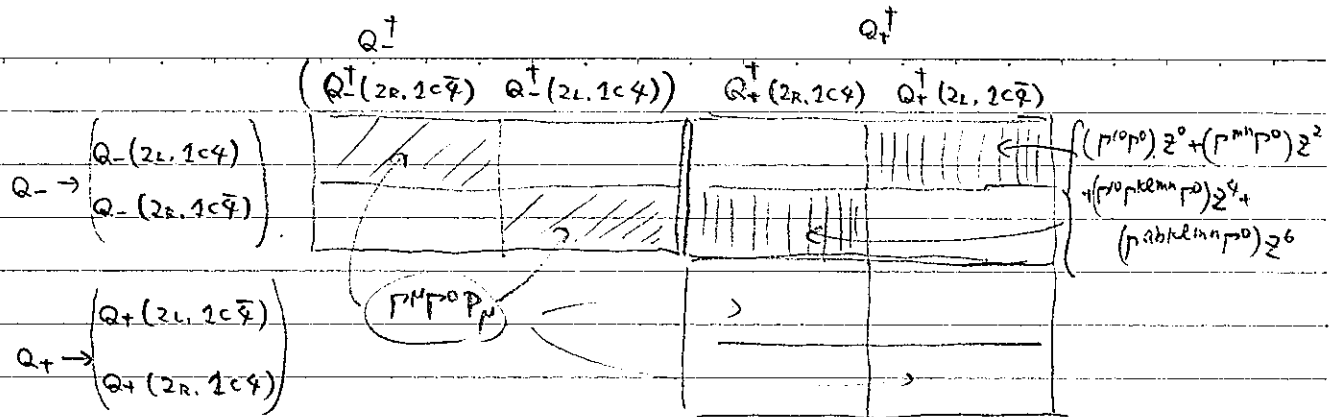
$$\text{add. } + \frac{1}{6!} (P^{ABKLMN})_{UR} Z_{ABKLMN}^{(6)}$$

Giveon Kutasov p.35 eq.(32)

Polchinski (B.2.9)

(contributions from
F1, NS5:
omitted.)

PLUS



read out

$\{Q_{\alpha}^2, \bar{Q}_{\beta,1}\}$	0	0	$\{Q_{\alpha}^1, \bar{Q}_{\beta}\}$
0		$\{\bar{Q}_{\alpha,1}, \bar{Q}_{\beta,2}\}$	0
		$\{Q_{\alpha}^2, \bar{Q}_{\beta,2}\}$	0
		0	

read out (ignore P^{10} and P^0)

$$Z_{4d} \sim \langle \downarrow\downarrow\downarrow | (*) | \downarrow\downarrow\downarrow \rangle$$

$$\begin{aligned}
 (*) = & T_{D0} N_{D0} + T_{D2} N_{D2,i} \int_{C''} (d\tilde{s}^1 d\tilde{s}^2) \frac{\partial \tilde{y}^{m_1}}{\partial \tilde{s}^1} \frac{\partial \tilde{y}^{m_2}}{\partial \tilde{s}^2} e_{m_1}^{a_1} e_{m_2}^{a_2} (\gamma^{a_1 a_2}) \\
 & + T_{D4} N_{D4} \int_{S''} d\tilde{s}^1 \dots d\tilde{s}^4 \frac{\partial \tilde{y}^{m_1}}{\partial \tilde{s}^1} \dots \frac{\partial \tilde{y}^{m_4}}{\partial \tilde{s}^4} e_{m_1}^{a_1} \dots e_{m_4}^{a_4} (\gamma^{a_1 a_2 a_3 a_4}) \\
 & + T_{D6} N_{D6} \int_M d\tilde{s}^1 \dots d\tilde{s}^6 \frac{\partial \tilde{y}^{m_1}}{\partial \tilde{s}^1} \dots \frac{\partial \tilde{y}^{m_6}}{\partial \tilde{s}^6} e_{m_1}^{a_1} \dots e_{m_6}^{a_6} (\gamma^{a_1 a_2 a_3 a_4 a_5 a_6})
 \end{aligned}$$

By using

$$\begin{cases}
 \gamma^{4,5} = \tau^{1,2} \otimes 1 \otimes 1 \\
 \gamma^{6,7} = \tau^3 \otimes \tau^{1,2} \otimes 1 \\
 \gamma^{8,9} = \tau^3 \otimes \tau^3 \otimes \tau^{1,2}
 \end{cases}$$

$$Z_{4d} \sim \left(N_{D0} - i N_{D2,i} \frac{\text{vol}(C'')}{l_s^2} - N_{D4} \frac{\text{vol}(S'')}{l_s^4} + i \frac{\text{vol}(M)}{l_s^6} N_{D6} \right) \times \frac{2\pi}{\log s}$$

D-brane charge $\Leftrightarrow (N_{D0}, N_{D2}, N_{D4}, N_{D6}) = \mathcal{V}$.

a pair of D-branes : stable coexistence if $\text{Arg}(Z(v_1)) = \text{Arg}(Z(v_2))$

instability $|Z(v_1)| + |Z(v_2)| \geq |Z(v_1 + v_2)|$

★ D_p and $D(p+2)$ or D_p and $D(p+6)$ not a stable config.

★ $D4$ and $D0$, $D6$ and $D2$ not a stable config.

$D4$ and $\overline{D0}$ $D6$ and $\overline{D2}$ coexist and stable.

in a flat spacetime

$$\begin{cases} \epsilon' = \Gamma^0 \epsilon & \& \epsilon' = \Gamma^{01234} \epsilon \Rightarrow \boxed{\epsilon = \Gamma^{1234} \epsilon} \\ \epsilon' = -\Gamma^0 \epsilon & \& \epsilon' = \Gamma^{01234} \epsilon \Rightarrow \boxed{\epsilon = -\Gamma^{1234} \epsilon} \end{cases}$$

on a background w/ δ parameters
special holonomy : the covariantly const. spinor
should be part of the δ parameters.

* include B-field rev.

for a D-brane system wrapped on holomorphic cycles
in a Calabi-Yau n -fold X ...

$$Z \approx \frac{2\pi}{l_s g_s} \int_X e^{-t} \wedge v = (-1)^n \frac{2\pi}{l_s g_s} \int_X e^t \wedge v^v$$

$$t := (B + iJ)/l_s^2 \quad 2\text{-form on } X$$

v^v : for $v \in H^{\text{even}}(X; \mathbb{Q})$ multiply $(-1)^k$ to the $2k$ -form component.

$$n=3 \quad (X = CY_3)$$

$$\mathcal{V}_{D6}(V_X) = \text{ch}(V_X) \Gamma_0(TX)$$

$$\mathcal{V}_{D4}(S', V_S) = (i\sigma)_* \left[\text{ch}(V_S) e^{\frac{1}{2}G(TS)} \frac{\sqrt{\hat{A}(TS)}}{\sqrt{\hat{A}(N_{S|X})_{\text{hol}}}} \right] = i\sigma_* \left[\text{ch}(V_S) \frac{\sqrt{\text{td}(TS)}}{\sqrt{\text{td}(N_{S|X})}} \right]$$

S' : surface in X V_S : bundle on S'

$$\mathcal{V}_{D2}(C, V_C) = (i\sigma)_* \left[\text{ch}(V_C) e^{\frac{1}{2}G_2(TC)} \right]$$

C : curve in X , V_C : bundle on C

$$\mathcal{V}_{D0} = [\text{pt}]$$

$$\text{td}(E) = 1 + \frac{1}{2} G_1(E) + \frac{G_2(E) + G_1(E)^2}{12} + \frac{G_1(E)G_2(E)}{24} + \dots$$

$$\hat{A}(E_{\text{hol}}) = 1 - \frac{1}{12} \text{ch}_2(E) + \left(\frac{1}{2 \cdot 12} (\text{ch}_2(E))^2 + \frac{1}{2^3 \cdot 15} \text{ch}_4(E) \right) + \dots$$

$$\Gamma_C(E) = \exp \left[\frac{-\gamma}{2\pi i} \text{ch}_1(E) + \frac{\zeta(2)}{(2\pi i)^2} \text{ch}_2(E) - \frac{2! \zeta(3)}{(2\pi i)^3} \text{ch}_3(E) + \frac{3! \zeta(4)}{(2\pi i)^4} \text{ch}_4(E) - \dots \right]$$

$$E = TX, X = CY_3 \Rightarrow \Gamma_C(TX) = \left[1 + \frac{1}{24} G_2(TX) - \frac{\zeta(3)}{(2\pi i)^3} G_3(TX) \right]$$

$$\mathbb{Z}(V_{D0}) \sim 1.$$

$$\mathbb{Z}(V_{D0}(C, V_C)) \sim \left[\left(-t - \frac{1}{2} K_C \right) \right]_C^{(rk)} + [C(V_C)]_C \quad \left[C(V_S) \cdot \left(-t - \frac{1}{2} K_{C_{VS}} \right) + (C(V_S)^2 - C(V_S)) \right]_S$$

$$\mathbb{Z}(V_{D0}(S, V_S)) \sim \left[\frac{t^2}{2} - \frac{1}{2} C(TS) \cdot t + \frac{C_2(TS) + 3C_1(TS)^2}{24} \right]_S^{(rk)} + \left[C_1(V_S) \cdot \left(-t - \frac{1}{2} K_S \right) + ch_2(V_S) \right]_S$$

$$\mathbb{Z}(V_{D0}(V_X)) \sim \left[-\frac{t^3}{3!} - t \cdot \frac{C(TX)}{24} - \frac{\zeta(B)}{(2\pi i)^3} C_3(TX) \right]_X^{(rk)} + \left[C_1(V_X) \left(\frac{t^2}{2} + \frac{C(TX)}{24} \right) \right]_X$$

$$+ [ch_2(V_X) \wedge (-t)]_X + [ch_3(V_X)]_X$$

$$= \left[-\frac{t^3}{3!} - t \cdot \frac{C(TX)}{24} - \frac{\zeta(B)}{(2\pi i)^3} C_3(TX) \right]_X + \mathbb{Z}(V_{D0}(C(V_X), 1))$$

$$+ (C_2(V_X) - C_1(V_X)^2) \cdot t + \frac{1}{2} (C(V_X) - C(V_X) C(V_X))_?$$

on $X = CY_3$

$$(rk \times DX \text{ on } S') + (N_0 \times D0) \longrightarrow (rk \times DX \text{ - branes on } S' \text{ w/ } V_S)$$

if $rk \cdot N_0 > 0$.

$$\mathbb{Z}(V_{D0}(S, V_S))$$

$$ch(V_S) = rk + 0 + N \cdot [pt] = -C_2(V_S)$$

$$[n=2 \quad (X=K3)]$$

$$V_{D0}(V_X) = ch(V_X) \left(\Gamma_0(TX) = \sqrt{\hat{A}(TX)} = \sqrt{td(TX)} = 1 + \frac{C(TX)}{24} \right)$$

$$V_{D0}(C, V_C) = (i_C)_* \left[ch(V_C) e^{\frac{1}{2} C(TC)} \right]$$

$$V_{D0} = [pt]$$

(DX and $D0$: the same as above.)

$$D0 \text{ in } DX \iff ch_2(V) > 0$$

$$\overline{D0} \text{ in } DX \iff ch_2(V) < 0 \quad (\text{masses just add up.})$$

(instanton # $\sim -ch_2$)

("Z" for old (1.1) SUSY algebra central charge)

$$(ch_3(V) = \frac{1}{6} C_1(V)^3 - \frac{1}{2} C_2(V) \cdot C_1(V) + \frac{1}{2} C_3(V))$$

★ An expression that is believed to be correct.

$$|Z_{\text{sd}}| = M_{\text{pl}} \sqrt{8\pi} \cdot |(v_z X^I + m^I F_I) e^{+\hat{K}/2}|$$

$$\hat{K} = -\ln \left(i (X^I \bar{F}_I - \bar{X}^I F_I) \right)$$

$$X^I = (1, t^i)^T \quad F_I = ((2 - t^i \partial_i) F, \partial_i F)^T$$

$$\left(\begin{aligned} \tau &= \frac{(B+iJ)}{2s^2} = D_I \tau^I \\ H^2(X; \mathbb{R}) &= \text{Span}\{D_I\} \end{aligned} \right)$$

$$F = \int_X \left(\frac{t^3}{3!} - \frac{C_2(TX) \wedge t}{24} - \frac{\zeta(3)}{(2\pi i)^3} \frac{C_2(TX)}{2} \right) - \frac{1}{2} a_{ij} t^i t^j + \frac{1}{(2\pi i)^3} \sum_p n_p \text{Li}_3(e^{2\pi i \langle p, t \rangle})$$

choose eg. $a_{ij} = \begin{cases} 1 & i < j \\ \frac{1}{2} \int_{D_i} c_i(TD_i) \cdot D_j & i < j \\ \frac{1}{2} \int_{D_j} c_j(TD_j) \cdot D_i & j < i \end{cases}$ (or mod. ± 2 if symmetric)

$$\left(M_{\text{pl}} \sqrt{8\pi} |e^{\hat{K}/2}| \underset{\text{large vol}}{\approx} \frac{M_{\text{pl}} \sqrt{8\pi}}{\sqrt{\int_X \frac{t^3}{3!}}} = \frac{M_{\text{pl}} \sqrt{8\pi}}{\sqrt{8 \cdot \text{vol}(X)/24}} = \frac{2\pi}{2s^2} \right)$$

$$(N_{D0}; -N_{D2,i}; N_{D6}, N_{D8}^2) \left[\begin{array}{l} 1 \\ t^i + g(t^i) - 1 \\ -\frac{t^3}{3!} - \frac{1}{24} C_2(TX) - \frac{\zeta(3)}{(2\pi i)^3} C_2(TX) \\ \int_X \frac{t^3}{3!} - \frac{1}{24} C_2(TX) \wedge t + \frac{C_2(TX) + 3C_2(TX)^2}{24} \end{array} \right]$$

$$= (N_{D0}, -N_{D2,i}; N_{D6}, N_{D8}^2) \left[\begin{array}{c|c|c} \begin{array}{c} 1 \\ -1 \\ 0 \\ 0 \end{array} & \begin{array}{c} \text{---} \\ \uparrow \\ \text{---} \end{array} & \begin{array}{c} 0 \\ 0 \\ 1 \\ 0 \end{array} \\ \hline \begin{array}{c} 1 \\ t^i \\ (2-t^i \partial_i) F \\ \partial_i F \end{array} \end{array} \right]$$

$$\tau_{D_0/TS_i} \left(a_{ij} - \frac{1}{2} \int_{D_i} c_i(TD_i) \cdot D_j \right)$$

$$(v_z, m^I)$$

II B / $(C\Gamma_3 = W)$ DB-branes on a slag $L \Rightarrow Z_{\text{sd}} = M_{\text{pl}} \sqrt{8\pi} |e^{\hat{K}/2} (v_z X^I + m^I F_I)|$

$$R = -\ln \left(i \int_W D_{\text{sd}} \bar{D}_W \right) \cdot (v_z X^I + m^I F_I) = \int_L^{(3,0)} D_W$$

$(v, m) \otimes H_3(W; \mathbb{R}) \quad (L \rightarrow L' \text{ by an } \text{SU}(5) \text{ rotat'n}) \Leftrightarrow \text{Ang}_g(Z(L)) = \text{Ang}_g(Z(L')) \quad \left[\begin{array}{l} \text{coexist PLUS} \\ \text{stable.} \end{array} \right]$