QFT II/QFT   homework I (Sep. 28, 2020)

- Reports on these homework problems are supposed to be submitted through the U Tokyo ITC-LMS. We request that the file name includes the problem number, such as II-1***.pdf or ****-IV-2-IX-1.jpeg. The ITC-LMS will show who had submitted the file (student ID and name), so the file name will not have to contain your name or ID number. (this instruction may be updated later)

1. **Harmonic oscillator energy levels in path integral formulation** [B]
   In a harmonic oscillator system with mass m and frequency ω (see the lecture note for more about the convention), path-integral can be used to derive

$$\langle q_f | e^{-iHT} | q_i \rangle = \sqrt{\frac{m\omega}{2\pi i \sin(\omega T)}} e^{iS_{\rm cl}}, \tag{1}$$

$$S_{\rm cl} = \frac{m\omega}{2} \frac{1}{\sin(\omega T)} \left[\cos(\omega T)(q_f^2 + q_i^2) - 2q_i q_f \right]. \tag{2}$$

   Derivation of the result above is found in the lecture note, and also in section 8? of the textbook by Feynman and Hibbs. Now, use the result above, and expand it into the form of

$$\langle q_f | e^{-iHT} | q_i \rangle = \sum_n \psi_n(q_f) \psi_n^*(q_i) e^{-iE_n T}, \tag{3}$$

   to find out the energy eigenvalues $\{E_n\}$ and their corresponding wavefunctions, for the ground state and the first excited state. [Derivation will be found in the Feynman Hibbs textbook, but it will be fun if you do it on your own.]

2. **Schwinger–Dyson equation** [B]
   Consider a quantum field theory described in terms of a field $\phi(x)$ and its canonical conjugate momentum $\pi(x)$, where the Hamiltonian density is $\mathcal{H}(\pi, \phi)$. We introduce discrete time slices and write down the partition function of this system as

$$Z = \int \prod_{k,\vec{x}} [d\pi_k(\vec{x}) d\phi_k(\vec{x})] \; \exp\left[-i \int d^d\vec{x} \left\{ (\Delta t) \mathcal{H}(\pi_k, \phi_{k-1}) + (\pi_{k+1} - \pi_k)\phi_k \right\} \right]. \tag{4}$$

   Now, with this notation, note the following identity,

$$0 = \int \prod_{k,\vec{x}} [d\pi_k(\vec{x}) d\phi_k(\vec{x})] \; \frac{\partial}{\partial \phi_j(\vec{y})} \left(e^{i\int dt[p\dot{q} - H]} \; \phi_{\ell_1}(\vec{y_1}) \cdots \phi_{\ell_*}(\vec{y}_*) \pi_{m_1}(\vec{z}_1) \cdots \pi_{m_*}(\vec{z}_*) \right), \tag{5}$$

which is trivial because the integrand is a total derivative. The right hand side of the identity above, on the other hand, can be written also as sum of terms where the derivative $\partial_{\phi_j(\vec{y})}$ acts of individual factors in the integrand. Using an approximation that

$$\frac{1}{\Delta t}\delta_{j\ell} \simeq \delta(t_j - t_\ell), \tag{6}$$

derive the Schwinger–Dyson equation

$$\begin{aligned} &\langle T\left\{ i\left(\frac{\partial \mathcal{H}(\pi_{j+1}(\vec{y}), \phi_j(\vec{y}))}{\partial \phi_j(\vec{y})} + \dot{\pi}_j(\vec{y}) \right) \phi_{\ell_1}(\vec{y_1}) \cdots \phi_{\ell_p}(\vec{y}_p)\pi_m(\vec{z}) \right\}\rangle \\ &= \sum_{a=1}^{p} \delta(t_j - t_{\ell_a})\delta^d(\vec{y} - \vec{y}_{\ell_a}) \, \langle \phi(t_{\ell_1}, \vec{y}_1) \cdots \check{\phi}(t_{\ell_a}, \vec{y}_a) \cdots \phi(t_{\ell_p}, \vec{y}_p)\pi(t_m, \vec{z})\rangle, \end{aligned} \tag{7}$$

where $\check{\phi}$ means that the designated operator should be omitted. [This equation indicates how the equation of motion of the action can be used in quantum mechanical calculations. This relation holds true for any combinations of a bra state and a ket state (because the derivation above is not affected by the choice of boundary conditions at $t_{\rm ini}$ and $t_{\rm fin}$ in the path integraion). See Peskin–Schroeder section 9.6 for a version in the Lagrangian formulation.]