QFT II/QFT homework II (Oct. 05, 2020)

- Reports on these homework problems are supposed to be submitted through the U Tokyo ITC-LMS. We request that the file name includes the problem number, such as II-1***.pdf or ****-IV-2-IX-1.jpeg. The ITC-LMS will show who had submitted the file (student ID and name), so the file name will not have to contain your name or ID number. (this instruction may be updated later)

1. **More about path integral of a fermionic system** [B]
In an ordinary (bosonic) quantum mechanical system, its partition function for its canonical ensemble is expressed by parh integral

$$Z = \int \mathcal{D}p\mathcal{D}q \exp\left[-\int_0^\beta d\tau \left(H(p,q) - ip\partial_\tau q \right) \right] \tag{1}$$

with the periodic boundary condition

$$q(t = -i\beta) = q(t = 0). \tag{2}$$

In this homework problem, let us work out the analogue for a two-state (fermionic) quantum mechanical system. In the lecture, we have learned how to express time evolution of wavefunctions of this two state system in term of path integral; following the notation used in the lecture, where a state $|0\rangle c_1 + |1\rangle c_1$ corresponds to a wavefunction $\Psi(\bar{\theta}) = c_0 + \bar{\theta} c_1$, evolution in the negative imaginary direction is given by

$$\Psi_{\rm fin}(\bar{\theta}_N) = \int d\bar{\theta}_{N-1} d\theta_{N-1} \cdots d\bar{\theta}_0 d\theta_0 e^{\left[\sum_{k=0}^{N} \bar{\theta}_{k+1}\theta_k - \sum_{k=0}^{N-1} \bar{\theta}_k\theta_k - (\Delta\tau)\sum_{k=1}^{N} H(\theta_{k-1}, \bar{\theta}_k)\right]} \Psi_{\rm in}(\bar{\theta}_0), \tag{3}$$

where $\Delta\tau = \beta/N$.

   (a) Now, by using $Z = {\rm tr}[e^{-\beta H}] = \langle 0|e^{-\beta H}|0\rangle + \langle 1|e^{-\beta H}|1\rangle$, verify that

$$Z = \int d\bar{\theta}_N d\bar{\theta}_{N-1} \cdots d\bar{\theta}_0 d\theta_0 e^{[\cdots]} (\bar{\theta}_N + \bar{\theta}_0). \tag{4}$$

   (b) For an arbitrary function $f(\bar{\theta}_N, \bar{\theta}_0)$ depending on Grassmann coordinates $\bar{\theta}_N$ and $\bar{\theta}_0$, verify that

$$\int d\bar{\theta}_N (\bar{\theta}_N + \bar{\theta}_0) f(\bar{\theta}_N, \bar{\theta}_0) = f(-\bar{\theta}_0, \bar{\theta}_0). \tag{5}$$

   [This means that the factor $(\bar{\theta}_N + \bar{\theta}_0)$ inserted in a Grassmann integral can be regarded as something like a delta function $\delta(\bar{\theta}_N + \bar{\theta}_0)$.]

(c) [not a problem] By combining both, we see that the partition function of the canonical ensemble of the two state (fermionic) quantum mechanical system is

$$Z = \int \mathcal{D}\bar{\theta}\mathcal{D}\theta \exp\left[\int d\tau \ \left((\partial_\tau \bar{\theta})\theta - H(\theta, \bar{\theta})\right)\right] \tag{6}$$

with the **anti**-periodic boundary condition

$$\bar{\theta}(t = -i\beta) = -\bar{\theta}(t = 0). \tag{7}$$

(d) Compute

$$\int d\theta d\bar{\theta} d\theta' d\bar{\theta}' \exp\left[(\bar{\theta}, \bar{\theta}') \begin{pmatrix} -m & p \\ p & -m \end{pmatrix} \begin{pmatrix} \theta \\ \theta' \end{pmatrix}\right]. \tag{8}$$

Contrary to the case with a boson, we see a positive power of $(p^2 - m^2)$.

(e) We have discussed the partition function $Z = \mathrm{tr}(e^{-\beta H})$ of a free boson. Write up a discussion in the case of a free Dirac fermion, using the two following equations:

$$\ln(Z) = 2 \sum_{n \in \mathbb{Z}} V_d \int \frac{d^d k}{(2\pi)^d} \ln\left(E_{\vec{k}}^2 + (2\pi T(n + 1/2))^2\right) \tag{9}$$

and

$$\ln(Z) = 4V_d \int \frac{d^d k}{(2\pi)^d} \left(\frac{\beta E_{\vec{k}}}{2} + \ln(1 + e^{-\beta E_{\vec{k}}})\right). \tag{10}$$

[remark: the first term $+\beta E/2$ has the sign opposite from that in the case of a scalar field. So, in a combination of four real scalar fields and one Dirac fermion (also in a combination of one complex scalar and one Weyl fermion), the first terms cancel. This is due to supersymmetry. There is no such cancellation among the second terms (thermal contributions), however.]

2. **Chemical potential** [B]

One can switch from the canonical ensemble of a quantum system to its grand canonical ensemble by modifying the Hamiltonian H to $H - \sum_i \mu_i N_i$, where the label i runs over (a subset of) all the conserved numbers (charges) of the system of one's interest. N_i is the Noether charge of a U(1) symmetry, and μ_i the corresponding chemical potential. In the case of the Standard Model of particle physics, for example, the lepton number and the baryon number are conserved charges.[1] When an effective theory with much lower energy scale is considered, the number of atoms of various kinds (labeled by i) may be conserved separately.

(a) **relativisistic boson case** Suppose that an effective theory with a relativistic complex boson is given by the Hamiltonian:

$$H = \int d^d x \; \left(\pi^* \pi + (\nabla \Phi)^* (\nabla \Phi) + V(|\Phi|^2)\right), \tag{11}$$

where π is the canonical conjugate momentum of a complex scalar field Φ^*. First, write down the Noether charge $N = \int d^3 x J^0$ in terms of π and Φ. Secondly, carry out the Gaussian integral with respect to π and π^* in

$$Z = \int \mathcal{D}\pi \mathcal{D}\pi^* \mathcal{D}\Phi \mathcal{D}\Phi^* \exp\left[-i \int dt (H - \mu N) + i \int d^{d+1} x \left(\pi(\partial_t \Phi^*) + \pi^* (\partial_t \Phi)\right)\right], \tag{12}$$

to see how the Lagrangian is modified.

(b) Verify that the path integral with the $e^{\mu N \beta} = e^{i \int dt \mu N}$ factor is equivalent to the path integral without the modification, but with the field redefinition $\Phi_{\rm orignl}(x,t) = e^{-i\mu t} \Phi_{\rm new}(x,t)$.

3. **Geometric Quantization and An Alternative Treatment of Spin** [C]
A single spin-1/2 degree of freedom—a two state system—can be described in the path integral formulation by using a Grassmann variable (as in the Week 2 lecture and also in homework II-1); a spin system with one spin-1/2 degree of freedom at each lattice point can be described by using a Grassmann field on the space-time. There is an alternative description to those systems, however; instead of Grassmann numbers/fields, we use bosonic degrees of freedom. This homework problem II-3 introduces this alternative description for a spin-j degree of freedom.[2]

[1] when the tiny neutrino masses and non-perturbative electroweak effects are ignored, to be more precise.

[2] References: Matthias Blau "Symplectic Geometry and Geometric Quantization" a lecture note in PDF available online,
ask Google for more, with the key word "geometric quantization".
This homework problem II-3 was inspired by N. Nagaosa's textbook, "Quantum Field Theory in Condensed Matter Physics," §2.5. and also by E. Fradkin's textbook, "Field Theories of Condensed Matter Systems," §5.2.

(a) Before working on quantum mechanics of a single spin-j degree of freedom, think of quantum mechanics of a single free particle moving freely on a 1-dimensional space with the coordinate q, the simplest quantum mechanical system. We usually use either one of the space representation or the momentum representation, where the Hilbert space is the normalizable wavefunctions of q alone, $\psi(q) \in \mathcal{H}_{\rm sp}$ (or of the momentum p alone, $\tilde{\psi}(p) \in \mathcal{H}_{\rm mm}$), where the canonical conjugate pair of observables q and p are represented as $(q\times)$ and $-i\partial_q$ on $\mathcal{H}_{\rm sp}$ (or as $i\partial_p$ and $(p\times)$ on $\mathcal{H}_{\rm mm}$). The isomorphism between the module of the operator algebra $\mathbb{C}[q,p]$, $\mathcal{H}_{\rm sp}$ and $\mathcal{H}_{\rm mm}$, is given by the Fourier transformation; that is a well-known story so far. Alternatively, one may introduce a complex coordinate $z = (p+iq)/\sqrt{2}$ on the phase space $M = \{(p,q) \in \mathbb{R}^2\}$, and think of the space of wavefunctions $\Psi(p,q)$ that satisfy

$$\bar{\partial}_{\bar{z}}\Psi := \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial p} + i\frac{\partial}{\partial q}\right)\Psi(p,q) = 0, \tag{13}$$

with the normalization condition[3]

$$\int_M dpdq\, |\Psi(p,q)|^2\, e^{-|z|^2} < \infty. \tag{14}$$

A claim is that the vector space of such Ψ's, $\mathcal{H}_{\rm kl}$, can be used as the Hilbert space $\mathcal{H}$, on which the operator q and p are represented as

$$q \longrightarrow \frac{1}{\sqrt{2}i}\left((z\times) - \frac{\partial}{\partial z}\right), \qquad p \longrightarrow \frac{1}{\sqrt{2}}\left((z\times) + \frac{\partial}{\partial z}\right). \tag{15}$$

i. To get started, verify that $[q,p] = i$.

ii. (not meant as a homework problem) It is said that there will be an isomoprhism between $\mathcal{H}_{\rm kl}$ and $\mathcal{H}_{\rm sp/mm}$ as modules of $\mathbb{C}[q,p]$. TW have not been able to construct such an isomorphism, or to find an appropriate literature so far.

All of the space representation, momentum representation, and the alternative (holomorphic) representation can be regarded as different variations of the same general procedure, geometric quantization, as we see in the following. Here, we have a symplectic manifold (M,ω); provided the integral of $\omega/(2\pi)$ over compact

[3] known as Bargman–Segal space

2-cycles of M are integral, we may well think of a line bundle L over M where the covariant derivative is $\nabla := d - iA$, $dA = \omega$, so $c_1(L) = \omega/(2\pi)$. Here, we write down a few key statements from the geometric quantization procedure and ask the students to verify them for the example above as a homework problem.

In geometric quantization, the Hilbert space of the quantum theory is set as the space of "normalizable" sections[4] $\Psi(p, q)$ of L over the entire phase space M subject to the constraints $\nabla_{X_{f,m}}\Psi = 0$ for ${}^\forall m \in M$, ${}^\forall X_{f,m} \in P_m$, for an appropriately chosen subbundle $P \subset TM^{\mathbb{C}}$; here is a list of properties that P needs to satisfy: the fibre vector space P_m is of complex n-dimensions for every $m \in M$ when $\dim_{\mathbb{R}} M = 2n$, isotropic under the symplectic form ω on M, and is closed within P_m itself under the Lie-bracket operation. Starting from a given classical symplectic geometry (M, ω), there may be multiple different choices $P \subset TM^{\mathbb{C}}$, and hence possibly of different quantum theories (in principle).

i. In the case of $M = \mathbb{R}^2$ with the symplectic form $\omega = dp \wedge dq = i dz \wedge d\bar{z}$, verify for the three different choices of $P \subset TM^{\mathbb{C}}$, (sp) $P_m = \mathbb{C}\partial_p \subset TM^{\mathbb{C}}_m$, (mm) $P_m = \mathbb{C}\partial_q \subset TM^{\mathbb{C}}_m$, and (kl) $P_m = \mathbb{C}\bar{\partial}_{\bar{z}} \subset TM^{\mathbb{C}}_m$, that P satisfies the property stated above (unless that is obvious to you; if obvious, skip this and move on). $\mathcal{H}_{\rm sp}$, $\mathcal{H}_{\rm mm}$, and $\mathcal{H}_{\rm kl}$ are indeed characterized by $\nabla_{X_{f,m}}\Psi = 0$ for those three different choices of $P \subset TM^{\mathbb{C}}$ (as we can choose $A = pdq$ for (sp), $A = -qdp$ for (mm), and $A = -i\bar{z}dz$ for (kl)).

ii. The observables q and p are represented on the space of $\Psi(p, q)$ under the following rule. Suppose $f = p$ or $f = q$, or their linear combinations. Then fix a tangent vector X_f on M by

$$\omega(X_f, -) = i df, \tag{16}$$

and set the representation of f on $\Psi(q, p)$ to be

$$O_f := X_f - i \langle X_f, A \rangle + f, \tag{17}$$

where $dA = \omega$. By using $A = pdq$, $A = -qdp$, and $A = -i\bar{z}dz$ for the three choices of $P \subset TM^{\mathbb{C}}$, verify indeed that $O_q = (q\times)$ and $O_q = -i\partial_q$ on $\mathcal{H}_{\rm sp}$, $O_q = i\partial_p$ and $O_p = (p\times)$ on $\mathcal{H}_{\rm mm}$, and $O_z = (z\times)$ and $O_{\bar{z}} = \partial_z$ on $\mathcal{H}_{\rm kl}$.

[4]Let us avoid stepping too much into the definition of the normalizablility here. If you are interested, explore for yourself.

(b) Let us now work on a quantum theory of a spin-j degree of freedom. An idea is to think of the spin observables $s_{x,y,z}$ as a classical angular momentum pointing on a sphere $M = S^2$ with the radius j, first, and then to apply the procedure of the geometric quantization described above.

i. To motivate what we will do from 3(b)ii below, observe that $s_z = \cos\theta$, $s_\pm = e^{\pm i\phi}\sin\theta$, and $[\cos\theta, \phi] = -i$ reproduce $[s_z, s_\pm] = \pm s_\pm$.

ii. Set $\omega = -\frac{k}{2} d(\cos\theta) \wedge d\phi$ as the symplectic form on $M = S^2 = \mathbb{C}P^1$. Verify that $\int_M \omega/(2\pi) = k$ (which means that $\omega/(2\pi) \in H^2(M;\mathbb{Z})$ if and only if $k \in \mathbb{Z}$).

iii. Verify what is non-trivial in the following statements (skip and move on, if trivial): As a connection A on $M = \mathbb{C}P^1$ such that $dA = \omega$, we can choose

$$\begin{cases} A = -\frac{k}{2}(\cos\theta - 1)d\phi, & \theta \neq \pi, \\ A = -\frac{k}{2}(\cos\theta + 1)d\phi, & \theta \neq 0. \end{cases} \tag{18}$$

Let $\psi = e_0\psi_0 = e_\pi\psi_\pi$ be a section of the line bundle L with $\nabla = d - iA$, and ψ_0 on $\theta \neq \pi$ and ψ_π on $\theta \neq 0$ its local trivialization. Then $\psi_0 = e^{ik\phi}\psi_\pi$.

iv. Let us introduce a complex coordinate on $\mathbb{C}P^1 \simeq S^2$ by using the stereographic projection: $z := \tan(\theta/2)e^{i\phi}$. Verify that the line bundle L is seen as a holomorphic vector bundle, by exploiting the non-unitary gauge transformation:

$$d - i\widetilde{A}^{\theta\neq\pi} = g_0^{-1}(d - iA^{\theta\neq\pi})g_0 = d - k\frac{\bar{z}dz}{1+|z|^2}, \qquad g_0 := \cos^k(\theta/2), \tag{19}$$

$$d - i\widetilde{A}^{\theta\neq 0} = g_\pi^{-1}(d - iA^{\theta\neq 0})g_\pi, \qquad g_\pi := \sin^k(\theta/2), \tag{20}$$

$$\psi_0 = g_0\tilde{\psi}_0, \quad \psi_\pi = g_\pi\tilde{\psi}_\pi, \qquad \tilde{\psi}_0 = z^k\tilde{\psi}_\pi, \tag{21}$$

$$d - i\widetilde{A}^{\theta\neq\pi} = (d - i\widetilde{A}^{\theta\neq 0}) - k\frac{dz}{z}. \tag{22}$$

Note that the transition function $\tilde{g}_{0\pi} = z^k$ for $\tilde{\psi}_0 = \tilde{g}_{0\pi}\tilde{\psi}_\pi$ is holmorphic. Note also that

$$\omega = ki\frac{dz \wedge d\bar{z}}{(1+|z|^2)^2}. \tag{23}$$

v. Let us choose $\mathbb{C}\bar{\partial}_{\bar{z}}$ as the subbundle $P \subset TM^{\mathbb{C}}$. It is convenient to choose $A = -i\bar{z}dz/(1+|z|^2)$ so that $dA = \omega$; that is because $\langle \bar{\partial}_{\bar{z}}, A \rangle = 0$ then. The condition $\nabla_{X_f}\Psi = 0$ is now read as Ψ being a holomorphic section of

L. It is known that the vector space of the holomorphic sections of L (with $\int_{\mathbb{C}P^1} c_1(L) = k$) is of $(k+1)$-dimensions (for $k \geq 0$); can you list up $(k+1)$ independent holomorphic sections of L?

vi. (this is not a part of this homework problem) The normalization condition is

$$\infty > \int_{\mathbb{C}P^1} \sin\theta d\theta d\phi \ \left(|\psi_0|^2 = |\psi_\pi|^2\right) = \int_{\mathbb{C}P^1} \frac{2|dzd\bar{z}|}{(1+|z|^2)^{2+k}} \ |\tilde{\psi_0}|^2, \qquad (24)$$

so all the holomorphic sections of L are normalizable.

vii. (this is not a part of this homework problem) To summarize, for a positive integer $k \in \mathbb{N}_{>0}$, we have chosen a symplectic manifold $(M \simeq S^2, \omega_k)$ where ω depends on k. The geometric quantization procedure is applied, with $\mathbb{C}\bar{\partial}_{\bar{z}} \subset TM^{\mathbb{C}}$ used, to obtain the $(k+1)$-dimensional Hilbert space.[5] The parameter k may be identified[6] with $2j$, where j is the spin. The path integral over the canonical conjugate pair, $\mathcal{D}p(t)\mathcal{D}q(t)$, will be $\mathcal{D}\cos\theta(t)\mathcal{D}\phi(t)$ then; the phase $i\int dt\dot{q}p - i\int dtH$ in the exponent of the path integral will involve $dt\dot{q}p = dqp = A$ (which is proportional to k, regardless of which gauge is used).

[5] More generally, the Hilbert space may turn out to be finite dimensional as a result of the geometric quantization, when the symplectic manifold (M, ω) is compact.

[6] would be nice if it is possible to apply the idea around eq. (16, 17) to say something like $O_{s_z} = z\partial_z - k/2$.....