QFT II/QFT homework III (Oct. 12, 2020)

- **At the head of your report, please write your name, student ID number and a list of problems that you worked on in a report (like "II-1, II-3, IV-2").**

1. **Final State Phase Space** [B]

   (a) **cross section 1**: Let us consider a process of a pair of relativisitc (approximately massless) particles collide at the center of mass energy $\sqrt{s}$, and scatter into a pair of non-identical particles. Suppose that this pair of final state particles have the same mass, m (imagine $e^+ + e^- \to \mu^+ + \mu^-$, for example). Verify that the cross section is

   $$d\sigma = \frac{d\varphi}{2\pi}\frac{d(\cos\theta)}{32\pi s}\beta|\mathcal{M}|^2, \tag{1}$$

   where (φ, θ) are the azimuthal angle and the scattering angle at the center of mass frame. [i.e., the initial state particles come along the z-axis, and the final state particles are moving out to $\vec{p} = |\vec{p}|(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$] β is the velocity $|\vec{p}|/E$ of the final state particles.
   (The primary lesson to take out from this problem is that the cross section has an overall dependence β/s due to the kinematics (final state phase space). To see how powerful this understanding is, visit the web page
   https://pdg.lbl.gov/2020/reviews/contents_sports.html
   and click on the last entry, "Plots of cross sections and related quantities", and look at Figure 52.2.)

   (b) **cross section 2**: Suppose now that the final state particles are (approximately) massless, for simplicity. Verify, then, that the expression above can also be written as

   $$d\sigma \simeq \frac{d\varphi}{2\pi}\frac{dt}{16\pi s^2}|\mathcal{M}|^2, \tag{2}$$

   where t is the Mandelstam variable of the 2-body $\longrightarrow$ 2-body scattering. (it is an option to skip this problem)

   (c) **decay rate 1** Let us think of a particle at rest with mass M decaying to a pair of non-identical (approximately) massless particles. Verify, then, that

   $$d\Gamma \simeq \frac{d^2\Omega}{4\pi}\frac{1}{16\pi M}|\mathcal{M}|^2. \tag{3}$$

(d) **decay rate 2** Suppose that a particle at rest with mass M decays to a pair of a particle with mass $M' = M - \Delta m$ and an approximately massless particle. Verify, when $0 < \Delta m \ll M$, that

$$d\Gamma \simeq \frac{d^2\Omega}{4\pi}\frac{\beta}{8\pi M}|\mathcal{M}|^2, \tag{4}$$

where β is the velocity of the massive particle in the final state. [A lesson: when a scattering/decay process is barely allowed kinematically, the measure over the final state phase space yields a positive power of $\beta \ll 1$, and hence the cross section/decay rate is suppressed.]

(e) **decay rate 3** Let us now think of a particle at rest with mass M decaying to three non-identical particles (e.g., $\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$). The final state phase space is of 5-dimensions, after exploting the energy-momentum conservation. When the decaying particle is a scalar, or with a spin but without a polarization, however, there is SO(3) symmetry of space ($\mathbb{R}^3$) rotation acting on the final state phase, and the integral over the 5-dimensional phase space is reduced to one on a 2-dimensional space. We can take, for example, (E_1, E_2)—the energy of two particles (e.g., ν_μ and $\bar{\nu}_e$) as the coordinates of the 2-dimensional space, and the integral is in the form of

$$d\Gamma = \frac{1}{(2\pi)^3 8M}|\mathcal{M}|^2 dE_1 dE_2. \tag{5}$$

Now, assume that all the three particles in the final states are approximately massless, and work out the region in the (E_1, E_2) space that is kinematically possible; the area should be $M^2/8$. So, when the matrix element $\mathcal{M}(E_1, E_2)$ does not have a particular structure (such as singularity),the total decay rate is something like

$$\Gamma \sim \frac{M}{(8\pi)^3}\left\langle |\mathcal{M}|^2 \right\rangle. \tag{6}$$

[The absence of structure in $|\mathcal{M}|^2$ is an appropriate assumption for $\mu \to e\nu\bar{\nu}$, but not quite for $t \to W + b + g$.]

(f) (not meant as a homework problem) Suppose that the matrix element $\mathcal{M}$ for a 2-body decay in (3) is approximately M times the matrix element $\mathcal{M}$ for a 3-body decay in (6). The 3-body decay rate is smaller then the 2-body decay rate by a factor $1/32\pi^2$ then.

(g) (not meant as a homework problem) If you are interested in learning more on kinematics (such as Dalits plot), you might think of visiting the web page referred to earlier, and download a review article "Kinematics (rev.)" from that page. Derivation of (5) is also found there.

2. **Fermi Surface, Hole Excitation, Friedel Oscillation** [B or C]

Consider a Lagrangian (and corresponding Hamiltonian) of non-relativisitic electron.

$$\mathcal{L} = \psi^\dagger \left[i\partial_t + \frac{1}{2m}\vec{\partial}\cdot\vec{\partial} \right] \psi, \tag{7}$$

$$H = \int d^3x\, \psi^\dagger \left[-\frac{1}{2m}\vec{\partial}\cdot\vec{\partial} \right] \psi. \tag{8}$$

ψ is a 2-component spinor field (i.e., in the 2-dimensional representation of the space rotation SO(3) symmetry group). With the creation and annihilation operators of states with a given momentum, the field operators are written as

$$\psi(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3} \sum_r \xi_r a_{\vec{p},r} e^{-iE_{\vec{p}}t + i\vec{p}\cdot\vec{x}}, \qquad \left\{ a_{\vec{p},r},\ a^\dagger_{\vec{q},s} \right\} = \delta_{r,s}(2\pi)^3\delta^3(\vec{p}-\vec{q}). \tag{9}$$

Here, $E_{\vec{p}} = |\vec{p}|^2/(2m)$, and ξ_r is a dimensionless 2-component spinor; one can take $\xi_{r=\uparrow} = (1,0)^T$ and $\xi_{r=\downarrow} = (0,1)^T$, for example.

[OK to skip (a–c) if trivial for you. [B] until (e), [C] to go beyond.]

(a) Verify that

$$H' \equiv H - \epsilon_F \int d^3x\, \psi^\dagger\psi = \int \frac{d^3p}{(2\pi)^3} \sum_r (E_{\vec{p}} - \epsilon_F)\, a^\dagger_{\vec{p},r} a_{\vec{p},r} + \text{const.} \tag{10}$$

(b) Verify that $\psi^\dagger\psi$ is the $\mu = 0$ component of the Noether current J^μ corresponding to the electron-number symmetry (phase rotation of ψ) in (7). [remark: Thus, $H' = H - \epsilon_F N_e$, where N_e is the electron number. ϵ_F is regarded as the chemical potential (Fermi energy).]

(c) (not a homework problem) Note that this system can be described by a Lagrangian

$$\mathcal{L}' = \psi^\dagger \left[i\partial_t + \frac{1}{2m}\vec{\partial}\cdot\vec{\partial} + \epsilon_F \right] \psi. \tag{11}$$

(d) Let us define $b_{\vec{p},r} \equiv a^\dagger_{-\vec{p},r}$ for $|\vec{p}|$ below the Fermi momentum p_F; $\epsilon_F = E_{p_F}$. Rewrite H' in terms of $a_{\vec{p},r}$ with $|\vec{p}| \geq p_F$ and $b_{\vec{p},r}$ with $|\vec{p}| < p_F$, and show that the state with all the levels below the Fermi surface filled is the ground state of this Hamiltonian H'. It will be easy to see that $b_{\vec{p}}$ and $b^\dagger_{\vec{p}}$ are the annihilation and creation operators of a hole.

(e) Use the expression (9) and the observation in (d) to verify that

$$\begin{aligned} G(\vec{x}, t; \vec{y}, t') &:= \langle 0| T \left\{ \psi(\vec{x}, t) \psi_I^\dagger(\vec{y}, t') \right\} |0\rangle \qquad (12) \\ &= \mathbf{1}_{2\times 2} \int \frac{d^3p}{(2\pi)^3} \Big(\Theta(t-t')\Theta(|p| - p_F) e^{-i(E_p - \epsilon_F)(t-t') + i\vec{p}\cdot(\vec{x}-\vec{y})} \\ &\qquad - \Theta(t'-t)\Theta(p_F - |p|) e^{-i(E_p-\epsilon_F)(t-t') - i\vec{p}\cdot(\vec{x}-\vec{y})} \Big). \end{aligned}$$

On the other hand, the path integral formulation allows one to conclude right away from the action (11) that

$$G(\vec{x}, t; \vec{y}, t') = \mathbf{1}_{2\times 2} \int d\omega \frac{d^3p}{(2\pi)^3} \frac{i e^{-i\omega(t-t') + i\vec{p}\cdot(\vec{x}-\vec{y})}}{\omega - E_p + \epsilon_F + i \mathrm{sgn}(E_p - \epsilon_F)\epsilon}. \qquad (13)$$

Verify that the former expression (in the operator formalism) is reproduced from the latter (in the path integral formalism) by integrating over ω.

(f) Verify that

$$\begin{aligned} & \int_{-\infty}^{+\infty} dt' \, \langle 0| T \left\{ \psi^\dagger \Gamma_x \psi(\vec{x}, t) \; \psi^\dagger \Gamma_y \psi(\vec{y}, t') \right\} |0\rangle \\ = & \int_{-\infty}^{+\infty} dt'' \, \langle 0| T \left\{ \psi^\dagger \Gamma_x \psi(\vec{r}, t'') \; \psi^\dagger \Gamma_y \psi(\vec{0}, 0) \right\} |0\rangle, \qquad (14) \end{aligned}$$

where $\vec{r} := \vec{x} - \vec{y}$ and $t'' = t - t'$, is equal to

$$(14) = -i \operatorname*{tr}_{2\times 2}[\Gamma_x \Gamma_y] \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \, f(q), \qquad (15)$$

with

$$
\begin{aligned}
f(q) &= 2\int \frac{d^3k}{(2\pi)^3} \frac{\Theta(|\vec{q}-\vec{k}|-p_F)\Theta(p_F-|\vec{k}|)}{(\omega_{\vec{q}-\vec{k}}-\omega_{\vec{k}})i}, & (16)\\
&= \int \frac{d^3k}{(2\pi)^3} \frac{\Theta(p_F-|\vec{k}|)-\Theta(p_F-|\vec{q}-\vec{k}|)}{(2m)^{-1}[q^2-2\vec{k}\cdot\vec{q}]}, & (17)\\
&= \int \frac{d^3k}{(2\pi)^3} \frac{2\Theta(p_F-|\vec{k}|)}{(2m)^{-1}[q^2-2\vec{k}\cdot\vec{q}]}, & (18)\\
&= \cdots = \frac{mk_F}{(2\pi)^2}\left\{1+\frac{4k_F^2-q^2}{4k_Fq}\ln\left|\frac{2k_F+q}{2k_F-q}\right|\right\}. & (19)
\end{aligned}
$$

Here, Γ_x and Γ_y are 2x2 matrices yet to be specified at this moment. This $f(q)$ is konwn as Lindhard function of a free electron gas (or of conduction band electrons). [Once you verify the equivalence between (14) and (15, 16), the rest—in between (16) and (19)—is not so much about QFT, but just a math exercise. It is an option to just accept that (16)=(19), and move on.]

(g) Suppose that the conduction band electrons (approximated by a free electron gas system) have interactions with other quantum mechanical degrees of freedom

$$
\Delta H = \sum_i g_i(\psi^\dagger\Gamma_i\psi)(\vec{r}_i,t)s_i, \tag{20}
$$

where s_i is an operator acting on a quantum mechanical degree of freedom localized at $\vec{r}_i$, Γ_i a constant dimensionless valued 2×2 matrix, and g_i is a coupling constant. The combination g_is_i has dimension of [energy $\times$ volume].

By setting

$$
\langle 0|T\left\{\exp\left[-i\int dt\Delta H\right]\right\}|0\rangle_{\psi,\psi^\dagger\ \text{system}} =: T\exp\left[-i\int dt\Delta H_{\text{eff}}\right], \tag{21}
$$

verify that the effective Hamiltonian contains

$$
\Delta H_{\text{eff}} \supset -\sum_{i<j} J_{ij}s_is_j; \qquad J_{ij} := \underset{2\times2}{\text{tr}}[\Gamma_i\Gamma_j]\int\frac{d^3q}{(2\pi)^3}\, f(q)e^{i\vec{q}\cdot\vec{r}_{ij}}. \tag{22}
$$

[This is known as RKKY interaction. To learn more, look up references with a keyword "Friedel oscillation" and "RKKY interaction".]