QFT II/QFT homework IV (Oct. 19)

- **At the head of your report, please write your name, student ID number and a list of problems that you worked on in a report (like "II-1, II-3, IV-2").**

1. **Fyenman rule of external lines** [B]

   (a) In the classroom, we did not take enough time to explain what the Feynman rule is supposed to be for external states. What are they supposed to be in the momentum space version? and why? It will be a good exercise—if this is the first time to learn the Feynman rule—to think about that in the case of relativisitic Weyl fermion, Dirac fermion (both particle and its anti-particle).

2. **Cluster Decomposition** [C (or E-7)]
   This problem is more an academic question than a practical question. So, this is only for students with plenty of time to spend (and brain power).[1]

   For the initial state α and a final state β, where

$$\alpha = \mu^-(\vec{p}_1) + \mu^+(\vec{p}_2), \tag{1}$$
$$\beta = e^-(\vec{p}_3) + \bar{\nu}_e(\vec{p}_4) + \nu_\mu(\vec{p}_5) + e^+(\vec{p}_6) + \nu_e(\vec{p}_7) + \bar{\nu}_\mu(\vec{p}_8), \tag{2}$$

   the S-matrix matrix element has the following decomposition

$$\begin{aligned} S_{\beta\alpha} &= (2\pi)^4\delta^4(p_3+p_4+p_5-p_1)(2\pi)^4\delta^4(p_6+p_7+p_8-p_2)i\mathcal{M}_{\beta^-\mu^-}i\mathcal{M}_{\beta^+\mu^+} \\ &+ (2\pi)^4\delta^4(\sum_{i=3}^{8} p_i - (p_1+p_2))i\mathcal{M}_{\beta\alpha} \end{aligned} \tag{3}$$

   so that $\mathcal{M}$'s do not have momentum conservation delta functions; $\beta^- = e^- + \bar{\nu}_e + \nu_\mu$ and $\beta^+ = e^+ + \nu_e + \bar{\nu}_\mu$. Such a decomposition based on subprocess momentum conservation is called the cluster decomposition. The matrix elements $\mathcal{M}_{\beta^-\mu^-}$ and $\mathcal{M}_{\beta^+\mu^+}$ describe the decay processes of μ^- and μ^+ that may take place completely independently from each other at any distant points in space and time [see discussions on "cluster decomposition" in QFT textbooks, if necessary].

[1] A student in the 2016 class asked a question, which I could not answer well back then. His question has been reformulated into the form of a report problem. I still do not have confidence in how to think of this problem. I am happy to discuss this question with you! The part (a) and (b) are not guaranteed to be a good lead in addressing this issue.

Let us go beyond what is typically written in textbooks and dig into subtlties associated with the cluster decomposition. Think of a μ^-–μ^+ collider experiment (not realized anywhere on earth yet); when the μ^- beam (also μ^+ beam) is boosted enough, its lifetime becomes long enough to do such an experiment.[2] Although $\mu^\mp$ particles in the colliding beams decay well before the beam reaches the colliding spot in a detector, they may also decay by chance as they pass through the detector. In this homework problem, let us focus on how to do theory computation in terms of the matrix elements $\mathcal{M}_{\beta\alpha}$, $\mathcal{M}_{\beta^-\mu^-}$ and $\mathcal{M}_{\beta^+\mu^+}$ for the $\alpha \to \beta$ process in the detector.[3]

(a) Suppose that the beam 1 (μ^-) [resp. beam 2 (μ^+)] proceeds in the $(1, 0, -\theta)$ direction [resp. $(-1, 0, -\theta)$ directon] in the $(x, y, z) \in \mathbb{R}^3$ coordinate system; $0 < \theta \ll 1$. The cross section of the beam in a (y, z) plane is (w, h_1) [resp. (w, h_2)], so the area is $A_1 = wh_1$ [resp. $A_2 = wh_2$]; let its flux be Φ_1 [resp. Φ_2]. We assume that $\mu^\pm$ in the two beams are both relativistic, so we use $v_i = c = 1$. Now, in this set-up, I would guess that the rate (event counts per unit time) of $\mu^- + \mu^+$ scattering (with a cross secton $d\sigma$) is

$$(\Phi_1 A_1)(\Phi_2 A_2)\frac{2}{w\theta}d\sigma; \tag{4}$$

on the other hand,

$$(\Phi_1 A_1)(\Phi_2 A_2)\frac{2}{w\theta}\Gamma^2(\Delta t_{\rm rslv})(\Delta \ell_{\rm rslv})^3 \tag{5}$$

is the rate of μ^- decay and μ^+ decay processes taking place at unresolvablly close distance in space and time. Here, Γ is the decay rate of μ^- and μ^+, and $\Delta t_{\rm rslv}$ [resp. $(\Delta \ell_{\rm rslv})$] is the resolution in the time [resp. the position in the detector] a decay took place. We assume that $\Delta \ell_{\rm rslv} \ll h_i, w$ for simplicity.[4]

Do you agree with (4) and (5)? or did you get a different expression? Can one say that either one of (4) and (5) is much smaller than the other for all similar situations in QFT (so such a question does not matter practically)?

[2] For a beam energy 10GeV, the decay length $\tau \times c \times \gamma$ for $\tau \sim 2\times10^{-6}$s, $c \sim 3\times10^8$m/s and $\gamma = E/m \sim 10^2$ becomes 10^5m. Particles in a beam is lost by 10^{-5} as the beam moves forward by 1m.

[3] An idea of μ^-–μ^+ collider has been considered by experts to probe physics beyond the Standard Model by using $\mathcal{M}_{\beta\alpha}$ with $\alpha = \mu^- + \nu^+$, but with the final state totally different from β in this hoemwork problem.

[4] In the decay $\mu^- \to e^-\bar{\nu}_e\nu_\mu$, it is not easy to determine the actual point of the decay precisely. We have just made a simplifying assumption, because we are addressing an academic question at this moment.

(b) In part (a), we have obtained a factor

$$\Gamma^2(\Delta t_{\rm rslv})(\Delta \ell_{\rm rslv})^3 + \int d\sigma \tag{6}$$

multiplied to a common factor. It makes sense to split the region of integration $\int d\sigma$ into two, the region of the 6-body final state configuration space where separate momentum conservation $\delta(p_3+p_4+p_5-p_1)\delta(p_6+p_7+p_8-p_2)$ holds within the accuracy of measurement of the momenta, and the region where the momentum is conserved only as a hole $\delta(\sum_{j=3}^{8} p_j - (p_1+p_2))$. The scattering process with $\int d\sigma$ in the former region cannot be distinguished from the simultaneous decay processes captured by the first term of (6).

Now, it is tempting to capture the combination (6) also in the following way: applying the general formula of cross sections for the two terms of the matrix elements in (3) combined.

$$\sigma \;\Rightarrow\; \frac{1}{(2E_1)(2E_2)2} \int d\Pi_{\beta^-} d\Pi_{\beta^+} (2\pi)^4 \delta^4(p_\beta - (p_1+p_2)) \\ \left| \mathcal{M}_{\beta\alpha} + (2\pi)^4 \delta^4(p_{\beta^-} - p_1) i \mathcal{M}_{\beta^-\mu^-} \mathcal{M}_{\mu^+\mu^+} \right|^2 ; \tag{7}$$

here, we have introduced notations, $p_{\beta^-} = p_3 + p_4 + p_5$, $p_{\beta^+} = p_6 + p_7 + p_8$, and $p_\beta = p_{\beta^-} + p_{\beta^+}$;

$$\int d\Pi_{\beta^-} = \int \frac{d^3 p_3}{(2\pi)^3}\frac{1}{2E_3}\frac{d^3 p_4}{(2\pi)^3}\frac{1}{2E_4}\frac{d^3 p_5}{(2\pi)^3}\frac{1}{2E_5}, \tag{8}$$

and $\int d\Pi_{\beta^+}$ is also defined similarly. The $|\mathcal{M}_{\beta\alpha}|^2$ contribution on the right hand side of (7) yields $\int d\sigma$ in (6). The $|\mathcal{M}_{\beta^-\mu^-}\mathcal{M}_{\beta^+\mu^+}|^2$ contribution on the right hand side of (7)

$$\frac{1}{2E_1}\int d\Pi_{\beta^-}(2\pi)^4\delta^4(p_{\beta^-} - p_1)|\mathcal{M}_{\beta^-\mu^-}|^2 \\ \frac{1}{2E_2}\int d\Pi_{\beta^+}(2\pi)^4\delta^4(p_{\beta^+} - p_2)|\mathcal{M}_{\beta^+\mu^+}|^2 \times \frac{(2\pi)^4\delta^4(p_{\beta^-} - p_1)}{2} \tag{9}$$

looks like the first term of (6), if one allows to replace the last factor $(2\pi)^4\delta(p_{\beta^-} - p_1)/2$ by $(\Delta t_{\rm rslv})(\Delta \ell_{\rm rslv})^3$.

..... but are we doing something senseible here? Is there a good argument for the unjustified replacement above? If all the above arguement is roughly right, then does that also imply that we should also add to (6) the interference terms?

(c) (regarded [E-7]) Discuss cancellation of IR divergence in this process / set-up.

3. **Time ordered perturbation theory** [C]

   (a) When we learn quantum mechanics for the first time, we learn that for a state with energy E_*, states with energy level E_n contribute to the perturbative correction to E_* by $\Delta E_* = \sum_n |V_{n*}|^2/(E_* - E_n)$, where $V_{n*} := \langle n|H|*\rangle$ is the perturbation term in the Hamiltonian. This is just one example of general phenomenon that quantum processes give rise to corrections inverse proportional to the virtuality $(E_* - E_n)$. How can computations using Feynman diagrams in quantum field theory be consistent with the general principle of quantum mechanics? Discuss in any way you like.

      i. It is OK to read related sections of some QFT textbooks, follow computations and fill gaps between the lines; you can take a photo-copy of your computation notes of that process and submit it as a report. For example, an explanation is found in pp.71–72 of a lecture note by G. Sterman, available in the following URL.
      http://arxiv.org/abs/hep-ph/9606312