QFT II/QFT homework V (Oct. 26, 2020)

- Reports on these homework problems are supposed to be submitted through the U Tokyo ITC-LMS. We request that the file name includes the problem number, such as II-1***.pdf or ****-IV-2-IX-1.jpeg. The ITC-LMS will show who had submitted the file (student ID and name), so the file name will not have to contain your name or ID number. (this instruction may be updated later)

1. **Quantization of constrained systems** [C or D] Read some appropriate materials and learn quantization of constrained sytems using Dirac bracket. Weinberg's text book volume 1, Chapter 7.6 (and the appendix to Chap.7) is an example. (or look up online) ... and how does that work in the following systems in practice? [category [C] for the (a, b); category [D] by going further.]

   (a) Dirac fermion with he Lagrangian $\mathcal{L} = \overline{\Psi}(i\gamma^\mu D_\mu - m)\Psi$. Introductory QFT text-books often choose to deal with the Hermitian conjugate $\Psi^\dagger$ of the field Ψ as the canonical conjugate field of Ψ, instead of dealing with $\Psi^\dagger$ as a field independent of Ψ so both Ψ and $\Psi^\dagger$ have their own canonical conjugate fields (as in the case of a QFT with a complex scalar field). What if we choose to deal with both Ψ and $\Psi^\dagger$ in the Dirac Lagrangian as independent field, and apply the Dirac bracket procedure for quantization?

   (b) photon $\mathcal{L} = -(1/4)F_{\mu\nu}F^{\mu\nu}$.

   (c) Chern–Simons action $\int d^3x\mathcal{L} = \kappa \int d^3x\ \epsilon^{\lambda\mu\nu}A_\lambda(\partial_\mu A_\nu)$ in 2+1-dimensional space-time; $\epsilon^{\mu\nu\lambda}$ is the totally anti-symmetric tensor with $\epsilon^{012} = +1$, and κ is a non-zero constant.

   (d) chiral scalar field $S = \int d^2x(\partial_\mu\phi)(\partial^\mu\phi)$ on 1+1 dimensional space-time, with a contraint $\partial_0\phi = -\partial_1\phi$. (such a field may arise as a chiral edge mode on the (1+1)-dimensional boundary of a condensed matter (2+1)-dimensional system, and is also used as a part of the formulation of Heterotic string theory)

   (e) any other systems of your interest.

2. **Spin dependence in the s-channesl scattering** [B]

   (a) We studied the $e^- + e^+ \to \mu^- + \mu^+$ cross section during the class. What if the final state μ^- and μ^+ particles were spin-0 (scalar) particles? Now we consider a

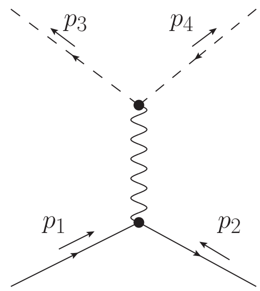

Figure 1: The Feynman diagram that contributes to the $e^-(p_1) + e^+(p_2) \to \tilde{\mu}^-(p_3) + \tilde{\mu}^+(p_4)$ scattering.

process of s-channel production of a pair of spin-0 particles ($\tilde{\mu}^-$ and $\tilde{\mu}^+$), whose Lagrangian is $\mathcal{L} = (D_\mu\Phi)^\dagger(D^\mu\Phi) - M^2|\Phi|^2$ using a complex scalar field Φ, instead of $\mathcal{L} = \overline{\Psi}_{(\mu)}(i\gamma^\mu D_\mu - M)\Psi_{(\mu)}$ using a 4-component spinor field $\Psi_{(\mu)}$. The bilinear part is $\mathcal{L}_0 = (\partial_\mu\Phi)^\dagger(\partial^\mu\Phi) - M^2|\Phi|^2$, whose canocnical quantization leads to

$$\Phi_I = \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left(a_{\vec{p}}e^{-i\underline{p}\cdot x} + b^\dagger_{\vec{p}}e^{i\underline{p}\cdot x}\right), \tag{1}$$

$$\Phi^\dagger_I = \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left(a^\dagger_{\vec{p}}e^{i\underline{p}\cdot x} + b_{\vec{p}}e^{-i\underline{p}\cdot x}\right), \tag{2}$$

where $a_{\vec{p}}$ and $a^\dagger_{\vec{p}}$ are annihilation and creation operators of a spin-0 $\tilde{\mu}^-$ particle, and $b_{\vec{p}}$ and $b^\dagger_{\vec{p}}$ those of its anti-particle (spin-0 $\tilde{\mu}^+$ particle). The remnant is

$$\mathcal{L}_{\text{int}} = ieQ_{(\mu)}A_\mu[(\partial^\mu\Phi^\dagger)\Phi - \Phi^\dagger(\partial^\mu\Phi)] + (eQ_{(\mu)})^2A_\mu A^\mu|\Phi|^2. \tag{3}$$

Now, to the $e^- + e^+ \to \tilde{\mu}^- + \tilde{\mu}^+$ process, the s-channel photon exchange diagram as in Figure 1 is the only contribution. Here is a question. Do you understand that the A_μ–$\tilde{\mu}^-$–$\tilde{\mu}^+$ vertex as in the Figure contributes a factor $-ieQ_{(\mu)}(p_3 - p_4)^\mu$, instead of the factor $-ieQ_{(\mu)}\gamma^\mu$ in the A_μ–μ^-–μ^+ vertex? [skip, if trivial]

(b) Verify that the scattering amplitude is given by

$$\begin{aligned} &i\mathcal{M}\left(e^-_r(\vec{p}_1) + e^+_s(\vec{p}_2) \to \tilde{\mu}^-(\vec{p}_3) + \tilde{\mu}^+(\vec{p}_4)\right) \\ &\qquad = \frac{i(Q_{(\mu)}Q_{(e)}e^2)}{(p_1+p_2)^2 + i\epsilon}(\overline{v}_s(\vec{p}_2)\gamma^\mu u_r(\vec{p}_1))(p_3 - p_4)_\mu. \end{aligned} \tag{4}$$

(c) Verify that the average of the |matrix element|^2 over the initial state spin is given by

$$\frac{1}{4}\sum_{r,s}|\mathcal{M}|^2 = \frac{(Q_{(\mu)}Q_{(e)}e^2)^2}{s^2}\left(+4m_{(e)}^2|\vec{p}_3|^2 - 8(\vec{p}_1\cdot\vec{p}_3)^2 + 4|\vec{p}_3|^2(s/2 - m_{(e)}^2)\right) \quad (5)$$

in the center of mass frame ($\vec{p}_2 = -\vec{p}_1$ and $\vec{p}_4 = -\vec{p}_3$).

(d) We now ignore the electron mass; $m_{(e)} \sim 0.5$ MeV is much smaller than the muon mass $M_{(\mu)} \sim 106$ MeV anyway. Verify, when the center of mass energy of the $e^- + e^+$ collision is just high enough for a $\tilde{\mu}^- + \tilde{\mu}^+$ pair creation ($|\vec{p}_1| \simeq E_{\vec{p}_1} \simeq M_{(\mu)}$ but $E_{\vec{p}_1} > M_{(\mu)}$), that

$$\frac{1}{4}\sum_{r,s}|\mathcal{M}|^2 \simeq \frac{(Q_{(\mu)}Q_{(e)}e^2)^2}{2}\beta_{(\mu)}^2\sin^2\theta, \quad (6)$$

where $\beta_{(\mu)} := |\vec{p}_3|/M_{(\mu)}$ is the velocity of the produced $\tilde{\mu}^\pm$, and θ the scattering angle in the center of mass frame ($\vec{p}_3\cdot\vec{p}_1 =: |\vec{p}_3||\vec{p}_1|\cos\theta$).

(e) Verify in the high-energy limit ($|\vec{p}_1| \gg M_{(\mu)}$) that

$$\frac{1}{4}\sum_{r,s}|\mathcal{M}|^2 \simeq \frac{(Q_{(\mu)}Q_{(e)}e^2)^2}{2}\sin^2\theta. \quad (7)$$

(f) (not as a part of the homework problem) The nearly massless initial state pair $e^- + e^+$ terns into a virtual photon with polarization transverse to the axis of the $e^- + e^+$ collision (lecture note for Week 05 and 06). This virtual photon couples to the p-wave relative wavefunction of the final state $\tilde{\mu}^- + \tilde{\mu}^+$ pair. This is why the pair of spin-0 particles are more likely to come out in the direction transverse ($\sin^2\theta \sim 1$ than ~ 0) to the axis of collision. The pair creation processes of spin-1/2 particles and spin-0 particles have different θ-dependence in their differential cross section (cf. Week 05 lecture). This can be used to determine the spin of pair-produced particles experimentally.

3. **Forward-backward Asymmetry** [B]

(a) Consider an s-channel process of $e^-(p_1) + e^+(p_2) \to V^* \to f^-(p_3) + f^+(p_4)$ going through a vector field V to a pair of spin-1/2 fermions $f^- + f^+$. This process is similar to the QED process $e^- + e^+ \to \gamma^* \to \mu^- + \mu^+$, but now we think of a theory where the vector field V_μ couples to the 4-component spinor fields for $e^\mp$ and $f^\mp$ by

$$\mathcal{L}_{\text{int}} = -g_L^{(i)}\overline{\Psi}_i\gamma^\mu V_\mu \left(\frac{1-\gamma_5}{2}\right)\Psi_i - g_R^{(i)}\overline{\Psi}_i\gamma^\mu V_\mu \left(\frac{1+\gamma_5}{2}\right)\Psi_i, \tag{8}$$

where $i = e, f$, and

$$\gamma_5 := \begin{pmatrix} -\mathbf{1}_{2\times 2} & \\ & \mathbf{1}_{2\times 2} \end{pmatrix}, \tag{9}$$

so $P_L = (1-\gamma_5)/2$ and $P_R = (1+\gamma_5)/2$ retain only the left-handed and right-handed spinor components, respectively.

Compute i) $|\mathcal{M}(e^-_\downarrow + e^+_\uparrow \to f^-_\downarrow + f^+_\uparrow)|^2$, and ii) $|\mathcal{M}(e^-_\downarrow + e^+_\uparrow \to f^-_\uparrow + f^+_\downarrow)|^2$ in the relativistic limit ($|\vec{p}_{1,2,3,4}| \simeq E$ in the center of mass frame and $m_{(e)}$ and $m_{(f)}$ negligible when compared with E). Here, the states $|n^\pm, \vec{p}, \uparrow\rangle$ and $|n^\pm, \vec{p}, \downarrow\rangle$ for $n = e, f$ are characterized by

$$\vec{p}\cdot\vec{s}|n^\pm, \vec{p}, \uparrow\rangle = +\frac{E}{2}|n^\pm, \vec{p}, \uparrow\rangle, \qquad \vec{p}\cdot\vec{s}|n^\pm, \vec{p}, \downarrow\rangle = -\frac{E}{2}|n^\pm, \vec{p}, \downarrow\rangle; \tag{10}$$

$\vec{s}$ is the spin operator.

(b) Draw a sketchy graph of those $|\mathcal{M}|^2$'s as a function of $\cos\theta \in [-1, +1]$; $\vec{p}_1 \cdot \vec{p}_3 =: |\vec{p}_1||\vec{p}_3|\cos\theta$.

(c) Even if the initial state $e^\pm$ are not polarized, and we do not detect the spin of the final state $f^\pm$, we may still define the forward-backward asymmetry

$$A_{FB} = \frac{\sigma(e^-e^+ \to f^-f^+ \mid \theta \le \pi/2) - \sigma(e^-e^+ \to f^-f^+ \mid \theta \ge \pi/2)}{\sigma(e^-e^+ \to f^-f^+ \mid \theta \le \pi/2) + \sigma(e^-e^+ \to f^-f^+ \mid \theta \ge \pi/2)}, \tag{11}$$

if it is possible to measure the charge of the final state particles (and distinguish f^- from f^+) experimentally ($A_{FB} > 0$ means that f^- comes out more often in the forward direction of the e^- beam than in the backward direction of the e^- beam).

Verify that A_{FB} is non-zero if $g_L^{(e)} \neq g_R^{(e)}$ and $g_L^{(f)} \neq g_R^{(f)}$.

(d) (this is not a problem) The coupling of the Z-boson in the Standard Model has this property indeed. The experimental data of $e^- + e^+ \to \mu^- + \mu^+$ shown in the Week 05–06 lectures clearly have a signal of the forward-backward asymmetry.[1] Further

[1] We should not use $|\mathcal{M}_{\text{photon}}|^2 + |\mathcal{M}_{\text{Z}}|^2$, but $|\mathcal{M}_{\text{photon}} + \mathcal{M}_{\text{Z}}|^2$ in computing the cross section, in fact. So, the standard model prediction for A_{FB} is not as simple a formula as what is obtained in part (c).

reading: Phys. Rept. **271** (1996) 181–266 “Experimental tests of the standard model in $e^+e^- \to f\bar{f}$ at the Z resonance” or arXiv:hep-ex/0509008 “Precision Electroweak Measurements on the Z Resonance.”