QFT II/QFT

homework VII (Nov. 9, 2020)

- Reports on these homework problems are supposed to be submitted through the U Tokyo ITC-LMS. We request that the file name includes the problem number, such as II-1***.pdf or ****-IV-2-IX-1.jpeg. The ITC-LMS will show who had submitted the file (student ID and name), so the file name will not have to contain your name or ID number. (this instruction may be updated later)

1. **Non-relativistic QED Lagrangian, Fine Structure of Hydrogen Spectrum** [C]

   Let us derive QED Lagrangian in the non-relativistic limit. We use the following conventions:

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & \\ & -\mathbf{1} \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} & \tau^i \\ -\tau^i & \end{pmatrix}, \tag{1}$$

   and $\eta^{\mu\nu} = \mathrm{diag}(+,-,-,-)$.

   (a) Now let us begin with the following QED Lagrangian (with $e > 0$ and $Q_e = -1$ in $D_\mu = \partial_\mu + ieQ_eA_\mu$),

$$\mathcal{L} = \overline{\Psi}\left[i\gamma^\mu\left(\partial_\mu - ieA_\mu\right) - m\right]\Psi = \Psi^\dagger(i\partial_t\Psi) - \Psi^\dagger H^{(0)}\Psi, \tag{2}$$

$$H^{(0)} = \begin{bmatrix} m - e\varphi & \vec{\tau}\cdot\left(-i\vec{\partial} + e\vec{A}\right) \\ \vec{\tau}\cdot\left(-i\vec{\partial} + e\vec{A}\right) & -m - e\varphi \end{bmatrix}. \tag{3}$$

   Here,

$$\Psi = \begin{pmatrix} \psi^{(0)} \\ \overline{\psi^c}^{(0)} \end{pmatrix} \tag{4}$$

   is a 4-component spinor field, and $\overline{\Psi} := \Psi^\dagger\gamma^0$, $A_\mu = (\varphi, -\vec{A})$. It is known that the upper 2 components $\psi^{(0)}$ is almost an electron field, and the lower 2 components $\psi^{c(0)}$ a positron field, when the gamma matrices are chosen as in (1). In the following, we will go through a series of field redefinition to make this distinction even more clearer. As a first step, define a new 4-component spinor field $\Psi^{(1)}$ by

$$\Psi^{(1)} := \exp\left[\begin{pmatrix} & \frac{\vec{\tau}\cdot(-i\vec{\partial}+e\vec{A})}{2m} \\ -\frac{\vec{\tau}\cdot(-i\vec{\partial}+e\vec{A})}{2m} & \end{pmatrix}\right]\Psi =: \exp\left[S_1\right]\Psi, \tag{5}$$

   and rewrite the Lagrangian (3) in terms of $\Psi^{(1)}$, rather than Ψ. Keep the terms that are $\mathcal{O}(m^{+1})$, $\mathcal{O}(m^0)$ and $\mathcal{O}(m^{-1})$ for now, and ignore the rest. Did you get

something like this? [OK to skip this, if you have done that before]

$$\mathcal{L} = \Psi^{(1)\dagger}(i\partial_t \Psi^{(1)}) - \Psi^{(1)\dagger} H^{(1)} \Psi^{(1)}, \tag{6}$$

$$H^{(1)} = \begin{bmatrix} m - e\varphi + \frac{(-i\vec{\partial}+e\vec{A})^2 + e\vec{B}\cdot\vec{\tau}}{2m} & \frac{-ie\vec{E}\cdot\vec{\tau}}{2m} \\ \frac{ie\vec{E}\cdot\vec{\tau}}{2m} & -m - e\varphi + \frac{-(-i\vec{\partial}+e\vec{A})^2 - e\vec{B}\cdot\vec{\tau}}{2m} \end{bmatrix} + \mathcal{O}(m^{-2}). \tag{7}$$

This is how we obtain the coupling (in the upper left matrix entry)

$$H \simeq m - e\varphi + \frac{(-i\vec{\partial}+e\vec{A})^2}{2m} + \Delta H, \qquad \Delta H = \frac{e}{m}\vec{B}\cdot\vec{s} =: \frac{g_{\rm tree} e}{2m}\vec{B}\cdot\vec{s} \quad \left(\vec{s} = \frac{\vec{\tau}}{2}\right) \tag{8}$$

of an electron, and the tree-level value $g_{\rm tree} = 2$.

(b) In order to get rid of the off-diagonal term that still exists at $\mathcal{O}(m^{-1})$, define

$$\Psi^{(2)} := \exp\left[\frac{-i}{4m^2}\begin{pmatrix} & \vec{\tau}\cdot e\vec{E} \\ \vec{\tau}\cdot e\vec{E} & \end{pmatrix}\right]\Psi^{(1)} =: \exp[S_2]\Psi^{(1)}; \tag{9}$$

here, $\vec{E} = -\partial_t \vec{A} - \vec{\partial}\varphi$. By rewriting the Lagrangian in terms of $\Psi^{(2)}$, show that

$$\mathcal{L} = \Psi^{(2)\dagger}(i\partial_t \Psi^{(2)}) - \Psi^{(2)\dagger} H^{(2)} \Psi^{(2)}, \tag{10}$$

$$\begin{aligned} H^{(2)} = {} & \begin{bmatrix} m & \\ & -m \end{bmatrix} - e\varphi \\ & + \frac{1}{2m}\left((-i\vec{\partial}+e\vec{A})^2 + \vec{B}\cdot\vec{\tau}\right)\begin{bmatrix} \mathbf{1} & \\ & -\mathbf{1} \end{bmatrix} \\ & + \frac{e(\vec{\partial}\cdot\vec{E})}{8m^2} + \frac{e}{4m^2}(\vec{E}\times(-i\vec{\partial}+e\vec{A}))\begin{pmatrix} \vec{\tau} & \\ & \vec{\tau} \end{pmatrix} + \cdots . \end{aligned} \tag{11}$$

Here, $\mathcal{O}(1/m^3)$ terms as well as off-diagonal terms of $\mathcal{O}(1/m^2)$ are omitted. Because of the absense of off-diagonal terms of $\mathcal{O}(1/m)$, the upper 2 components of $\Psi^{(2)}$, $\psi^{(2)}$, is more purely an electron field than $\psi^{(0)}$ is. [remark: although there are off-diagonal terms $\psi^{(2)}\psi^{c(2)}$ whose coefficients are of $\mathcal{O}(1/m^2)$, they can be absorbed by further field redefinition. After the redefinition, diagonal terms of $\mathcal{O}(1/m^3)$ will appear. In that sense, the off-diagonal $\mathcal{O}(1/m^2)$ terms are smaller effects than the diagonal $\mathcal{O}(1/m^2)$ effects.] Because $\vec{E} = \frac{e}{4\pi}\frac{\vec{r}}{r^3}$ around a proton, the last term gives rise to a spin–orbit coupling of the form $\frac{\alpha}{4m^2 r^3}\vec{L}\cdot\vec{\tau}$.

(c) (a remark: not a homework problem) You might have encountered this $\vec{L}\cdot\vec{S}$ coupling as a quantum mechanics exercise on "addition of spin." Now, you understand where it comes from! See also the remark in part (e).

(d) (if you are not tired yet...) Although we have carried out $1/m$-expansion of the QED Lagrangian so far, we should be aware that the expansion parameter $1/m$ is not a dimensionless constant. The $1/m$ corrections always come in the form of a dimensionless combination (E/m) for some energy scale E, and the $1/m$ expansion becomes useful only when this ratio (E/m) is small. Thanks to the fact that the Bohr radius is $r_B = 1/(m\alpha)$ in natural unit, and the fact $\alpha \simeq 1/137 \ll 1$, this condition is actually satisfied in condensed matter applications (where $|\vec{p}| = |-i\vec{\partial}| \lesssim 1/r_B$ (Brillouin zone)), and also in atomic physics (where $|\vec{p}| \sim 1/r_B$); the ratio $|\vec{p}|/m \sim \alpha \sim 1/137$ actually plays the role of expansion parameter.

The hydrogen binding energy comes from $e\varphi \sim \alpha/r_B \sim m\alpha^2$ and $p^2/m \sim 1/(mr_B^2) \sim m\alpha^2$. Show that the last two terms of (11) give rise to the correction of $\mathcal{O}(m\alpha^4)$ to the binding energy by naive analysis as above.

In order to determine the $\mathcal{O}(m\alpha^4)$ correction to the binding energy of a hydrogen atom, however, one will notice that the p^4/m^3 term also needs to be calculated. Because this term is not generated by the field redefinition that absorbs the $\mathcal{O}(1/m^2)$ off-diagonal terms in (11), one only needs to calculate the p^4/m^3 term in the Lagrangian (11) written in terms of $\Psi^{(2)}$. Show that it is

$$\Delta\mathcal{L} = \Psi^{(2)\dagger}\left[\frac{(\vec{p}^2)^2}{8m^3}\begin{pmatrix}\mathbf{1} & \\ & -\mathbf{1}\end{pmatrix}\right]\Psi^{(2)}. \tag{12}$$

(e) (not a homework problem: just for your fun) Therefore, the fine structure of a hydrogen atom can be calculated by treating

$$\Delta H = -\frac{(\vec{p}^2)^2}{8m^3} + \frac{e(\vec{\partial}\cdot\vec{E})}{8m^2} + \frac{e}{4m^2}\vec{\tau}\cdot(\vec{E}\times\vec{p}) \tag{13}$$

as additional part of the Hamiltonian of Schrödinger equation, and evaluating the matrix element of this ΔH using the wavefunctions labeled by quantum numbers (n, l, m). Here, $\vec{p} = -i\vec{\partial}$, $e\varphi(r) = \alpha/r$, and $\vec{E} = -\vec{\partial}\varphi$. See [Landau–Lifshtz series vol 4, Qauntum Electrodynamics section 33–34]. The binding energy at order $\mathcal{O}(m\alpha^4)$ becomes

$$E_{n,j,l,s} \simeq m\left(-\frac{\alpha^2}{2n^2} - \frac{\alpha^4}{2n^3(j+1/2)} + \frac{3}{8}\frac{\alpha^4}{n^4}\right). \tag{14}$$

The degeneracy among the states with the same value of n is partially lifted, although the states with the same n and the same j (that is, $(l,s) = (j-1/2, 1/2)$ state and $(l,s) = (j+1/2, 1/2)$ state) are still degenerate at this $\mathcal{O}(m\alpha^4)$ level.

The same result can be obtained also by solving Dirac equation with $\varphi(r) = e/(4\pi r)$, instead of the Schrödinger equation. The result is this:[1]

$$E = m\left(1 + \left(\frac{\alpha}{n-(j+1/2)+\sqrt{(j+1/2)^2-\alpha^2}}\right)^2\right)^{-1/2}. \qquad (15)$$

Expanding this expression as a power series of α, (14) is obtained. Note that the degeneracy between the $l = j \mp 1/2$ states is not lifted at all order in α in this expression.

This remaining degeneracy is lifted by a QED 1-loop effect (Lamb shift),[2] which is not taken into account by simply solving the Dirac equation.

(f) (a remark, not a homework problem) The discussion in this homework problem should be read as that for a hydrogen atom; a simple generalization $\varphi = (Ze)/(4\pi r)$ enables us to deal with a hydrogen-like atom, where the nucleus has $+Ze$ charge and one electron $-e$ charge.

In principle, one may consider recycling this same discussion to deal with an alkali atom (Li, Na, K, Rb etc), where we make an approximation that innter shell $(Z-1)$ electrons shield the $+(Z-1)e$ electric charge of the nucleus and focus on the spectrum of the remaining one electron. TW does not know how good this approximation is when it comes to making a quantitative prediction.

When it comes to a non-alkali atom, electron-electron Coulomb interaction is essential in determining the atomic spectrum. Imagine such atoms as C or O, for example. The Coulomb potential between the outer shell electrons contribute to the Hamiltoninan (energy) by $\alpha/r_B \sim m\alpha^2$, while the L–S coupling that we deal with in this homework problem comes at $\mathcal{O}(m\alpha^4)$. An empirical rule in determining the spectrum of such an atom is summarized in Hund's rule;[3] the electron–electron Coulomb repulsion (which mimics S–S coupling due to the

[1] See Landau–Lifshitz vol. 4 (QED) section 36, or K. Nishijima, "*Relativistic Quantum Mechanics*", (Baifu-Kan).

[2] See Bjorken–Drell section 8.7 (or Landau–Lifshitz vol 4 (QED) section 123).

[3] for example, J. Kondo, "*Kinzoku-denshi-ron*," sections 1.2 and 1.3. Shokabo Publ., *butsurigaku Sensho* series. (this book is in Japanese)

exchange interaction) is treated with higher priority than the L–S coupling in Hund's rule.

The L–S coupling does not just give rise to fine structure in the spectrum of an isolated atom. When a bulk material is treated by the tight binding model (an eleectron state at atomoic energy levels localized at an ion site along hopping between those states of neighbouring ions), atomic energy levels—including consequences of the L–S couplings—are used as an input. For large Z atoms, the L–S coupling is not particularly negligible compared with the electron–electron Coulomb repultions, probably because the coefficient of the L–S coupling involves the epectation value of $\langle E \rangle$ and the electric potential from the ion is not completely shielded by the inner shell electrons at finite radius.

2. **Scattering amplitudes with interfering terms** [B]

(a) To the $e^- + e^+ \to e^- + e^+$ scattering (Bhabha scattering), there are two contributions to the scattering amplitude at tree level. One corresponds to a Feynman diagram where a photon is exchaned in the s-channel (as in $e^- + e^+ \to \mu^- + \mu^+$), and the other to a diagram where a photon is exchanged in the t-channel (as in $e^- + \mu^+ \to e^- + \mu^+$). Verify that

$$\frac{1}{4} \sum_{r,s,r',s'} |\mathcal{M}|^2 \simeq 2 \left[\frac{t^2 + u^2}{s^2} + \frac{s^2 + u^2}{t^2} + \frac{2u^2}{st} \right], \tag{16}$$

and hence

$$\frac{d\sigma}{d\cos\theta} \simeq \frac{\pi\alpha^2}{s} \left[\sin^4(\theta/2) + \cos^4(\theta/2) + \frac{1 + \cos^4(\theta/2)}{\sin^4(\theta/2)} + \frac{2\cos^4(\theta/2)}{\sin^2(\theta/2)} \right], \tag{17}$$

where θ is the scattering angle in the center-of-mass frame; here, we assume that $\sqrt{s} \gg m$ and the electron mass is negligible.

(b) To the $e^- + e^- \to e^- + e^-$ scattering (Moeller scattering), there are two contributions to the scattering amplitude at tree level. One corresponds to a Feynman diagram where a photon is exchaned in the t-channel, and the other to a diagram with a photon exchange in the u-channel. Verify that

$$\frac{1}{4} \sum_{r,s,r',s'} |\mathcal{M}|^2 \simeq 2 \left[\frac{t^2 + s^2}{u^2} + \frac{s^2 + u^2}{t^2} + \frac{2s^2}{ut} \right]. \tag{18}$$

One can see that the matrix elements of the two processes satisfisfy the crossing symmetry.

3. **Scattering with a three-particle final state** [C]

Let us study the two-body to three-body scattering process $e^-(p_1)+e^+(p_2)\to\mu^-(p_3)+\mu^+(p_4)+V(p_5)$, where we assume that V is a massless vector field that couples only to $\mu^\pm$; this is meant to be a preliminary exercise for the process $e^-+e^+\to q+\bar{q}+g$. We assume that the center of mass energy is much larger than m_e or m_μ, so we deal with $e^\pm$ and $\mu^\pm$ as approximately massless particles. The scattering amplitude is then given by (covariant derivative is $D_\mu=\partial_\mu+ieQA_\mu+ig\widetilde{Q}V_\mu$)

$$\begin{aligned} i\mathcal{M}\simeq&\frac{+i(ieQ_{(e)})(ieQ_{(\mu)})(ig\widetilde{Q}_{(\mu)})}{s+i\epsilon}\left[\overline{v}_{s_2}(\vec{p}_2)\gamma_\mu u_{s_1}(\vec{p}_1)\right]\ \epsilon^*_\lambda(\vec{p}_5)_{s_5}\\ &\times\left\{\overline{u}_{s_3}(\vec{p}_3)\gamma^\lambda\frac{i[\not{p}_3+\not{p}_5]}{(p_3+p_5)^2}\gamma^\mu v_{s_4}(\vec{p}_4)+\overline{u}_{s_3}(\vec{p}_3)\gamma^\mu\frac{i[-\not{p}_4-\not{p}_5]}{(p_4+p_5)^2}\gamma^\lambda v_{s_4}(\vec{p}_4)\right\}. \end{aligned}\tag{19}$$

After a bit of work, it is possible to verify that

$$\frac{1}{4}\sum_{s_1,s_2,s_3,s_4,s_5=1}^{2}|\mathcal{M}|^2\simeq\frac{2e^4g^2(Q_{(\mu)}\widetilde{Q}_{(\mu)})^2\left\{(p_1\cdot p_3)^2+(p_1\cdot p_4)^2+(p_2\cdot p_3)^2+(p_2\cdot p_4)^2\right\}}{(p_1\cdot p_2)\,(p_3\cdot p_5)\,(p_4\cdot p_5)}.\tag{20}$$

In the case of $e^-+e^+\to q+\bar{q}+g$, we only need to do the following replacement:

$$(Q_{(\mu)}\widetilde{Q}_{(\mu)})^2\to Q^2_{(q)}\times\left[C_2N_c=\frac{4}{3}\times3=8\times\frac{1}{2}=4\right].\tag{21}$$

(a) Let us get the feeling of how (20) is derived from (19). Since there are two terms in $\{\cdots\}$ of (19), there are four terms in $|\mathcal{M}|^2$; we begin by focusing on the term obtained by $|\text{first term}|^2$. Apart from the obvious overall factor

$$\frac{e^4g^2(Q_{(e)}Q_{(\mu)}\widetilde{Q}_{(\mu)})^2}{s^2}\tag{22}$$

that is common to all the four terms, we have a factor

$$\frac{\sum_{s_5}\frac{1}{4}\mathrm{Tr}_{4\times4}\left[\gamma_\mu\not{p}_1\gamma_\nu\not{p}_2\right]\mathrm{Tr}_{4\times4}\left[\gamma^\kappa(\not{p}_3+\not{p}_5)\gamma^\mu\not{p}_4\gamma^\nu(\not{p}_3+\not{p}_5)\gamma^\lambda\not{p}_3\right]\epsilon^*_\kappa(\vec{p}_5)_{s_5}\epsilon_\lambda(\vec{p}_5)_{s_5}}{(2p_3\cdot p_5)^2}\tag{23}$$

in the $|\text{first term}|^2$ contribution. The sum over the spin (helicity) of the final state vector field V is carried out by

$$\sum_{s_5}\epsilon^*_\kappa(\vec{p}_5)_{s_5}\epsilon_\lambda(\vec{p}_5)_{s_5}\to-\eta_{\kappa\lambda},\tag{24}$$

just like $\sum_s u_s(\vec{p})\overline{u}_s(\vec{p}) = \not{p} + m$. [see Peskin–Shcroeder section 5.5 for why this is OK] Now, the numerator of (23) is

$$\begin{aligned} & -\frac{1}{4}\mathrm{Tr}_{4\times4}\left[\gamma_\mu \not{p}_1 \gamma_\nu \not{p}_2\right] \mathrm{Tr}_{4\times4}\left[\gamma^\kappa(\not{p}_3+\not{p}_5)\gamma^\mu \not{p}_4 \gamma^\nu(\not{p}_3+\not{p}_5)\gamma^\lambda \not{p}_3\right]\eta_{\kappa\lambda} \qquad (25) \\ = & -\left[(p_1)_\mu(p_2)_\nu + (p_2)_\mu(p_1)_\nu - (p_1\cdot p_2)\eta_{\mu\nu}\right] \\ & \qquad \mathrm{Tr}_{4\times4}\left[(\not{p}_3+\not{p}_5)\gamma^\mu \not{p}_4 \gamma^\nu(\not{p}_3+\not{p}_5)\gamma_\kappa \not{p}_3 \gamma^\kappa\right]. \end{aligned}$$

To proceed further, verify the following relations:

$$\gamma^\mu\gamma^\kappa\gamma^\nu + \gamma^\nu\gamma^\kappa\gamma^\mu = 2\eta^{\mu\kappa}\gamma^\nu - 2\eta^{\mu\nu}\gamma^\kappa + 2\eta^{\nu\kappa}\gamma^\mu, \quad \gamma^\mu\gamma^\kappa\gamma_\mu = -2\gamma^\kappa, \quad \not{p}\not{p} = p^2\mathbf{1}_{4\times4}. \qquad (26)$$

These relations (as well as $p_i^2 \simeq 0$ for $i = 1, \cdots, 5$) can be used to see that

$$\begin{aligned} (25) =& 2\mathrm{Tr}_{4\times4}\left[(\not{p}_3+\not{p}_5)\not{p}_3(\not{p}_5+\not{p}_3)\left(\not{p}_1\not{p}_4\not{p}_2 + \not{p}_2\not{p}_4\not{p}_1\right)\right] \\ & - 2(p_1\cdot p_2)\mathrm{Tr}_{4\times4}\left[(\not{p}_3+\not{p}_5)\not{p}_3(\not{p}_5+\not{p}_3)\gamma^\mu\not{p}_4\gamma_\mu\right], \\ =& 2\mathrm{Tr}_{4\times4}\left[\not{p}_5\not{p}_3\not{p}_5\left(\not{p}_1\not{p}_4\not{p}_2 + \not{p}_2\not{p}_4\not{p}_1\right)\right] \\ & - 2(p_1\cdot p_2)\mathrm{Tr}_{4\times4}\left[\not{p}_5\not{p}_3\not{p}_5\gamma^\mu\not{p}_4\gamma_\mu\right], \\ =& 4\mathrm{Tr}_{4\times4}\left[\not{p}_5\not{p}_3\not{p}_5\left(\not{p}_2(p_1\cdot p_4) + \not{p}_1(p_2\cdot p_4)\right)\right], \\ =& 32(p_1\cdot p_4)(p_5\cdot p_3)(p_5\cdot p_2) + 32(p_2\cdot p_4)(p_5\cdot p_3)(p_5\cdot p_1). \qquad (27) \end{aligned}$$

Factoring out $(p_3\cdot p_5)$ and cancelling it against the same factor in the denominator, we have

$$(23) = \frac{8(p_4\cdot p_5)\left((p_1\cdot p_4)(p_2\cdot p_5) + (p_2\cdot p_4)(p_1\cdot p_5)\right)}{(p_3\cdot p_5)(p_4\cdot p_5)}. \qquad (28)$$

By using the momentum conservation $p_5^\mu = (p_1+p_2-p_3-p_4)^\mu$ for all the p_5's in the numerator, one will find that the numerator contains

$$8(p_1\cdot p_2)\left((p_1\cdot p_4)^2 + (p_2\cdot p_4)^2\right) \qquad (29)$$

along with other terms. After cancelling the factor $(p_1\cdot p_2)$ with the same factor in $s = 2p_1\cdot p_2$ in the denominator, we have managed to recover two terms out of four in (20). Almost the same procedure for the |second term|^2 contribution recovers the two remaining terms in (20). Remaining terms in the |first term|^2 and |secont term|^2 contributions cancel against the mixed terms, although it is necessary to be aware of the following:

(b) In a scattering amplitude where total of N massless particles come in or go out, there are $3N-10$ Lorentz invariant kinematical variables. Certainly in the $N=4$ case (2-body to 2-body scattering), there are $3\times 4-10=2$ independent Mandelstam variables; there is one relation on s, t and u. The $(3N-10)$ degrees of freedom correspond to $\vec{p}_i$'s for $i=1,\cdots,N$ that are subject to four-momentum conservation (-4) and Lorentz transformation (-6). Now, in the $N=5$ case, as in this homework problem, there should be $3\times 5-10=5$ Lorentz invariant kinematica variables. If we take $\{p_i\cdot p_j \mid 1\le i<j\le 4\}$, there must be one more relation on those six kinemtaical variables. It is

$$(p_1\cdot p_2)-(p_1\cdot p_3)-(p_1\cdot p_4)-(p_2\cdot p_3)-(p_2\cdot p_4)+(p_3\cdot p_4)=0, \qquad (30)$$

when all the five particles are (approximately) massless. How can you derive it? [By rewriting the numerator of all the contributions in the spin-summed/averaged $|\mathcal{M}|^2$ only in terms of $(p_i\cdot p_j)$ with $1\le i<j\le 4$, and further eliminating $p_3\cdot p_4$ by using the relation above in terms of five others, one will be able to verify (using Mathematica, for example) that the cancellation referred to above takes place indeed.]

(c) (Remark, not a problem) Note that there are only four independent kinematical variables that depend on the momentum of final state particles, since $p_1\cdot p_2=s/2$ depends only on the momenta of the initial state particles. On the other hand, the integral over the final state phase space is over five dimensions, four-momentum conservation is imposed on $\prod_{i=3}^{5}[d^3\vec{p}_i]$. This discrepancy is due to the fact that we have focused on the spin averaged scattering cross section; with the spin average on the initial e^-+e^+, there is no specific direction in the azimuthal angle around the axis of e^-–e^+ collision, and hence the spin-averaged matrix element should have rotation symmetry around this axis.

(d) Let us use the center of mass frame for convenience, in an attempt to extract physics out of the result of QFT computation (20). Now, we use the following parametrization:

$$\vec{p}_1=E\hat{e}_z=-\vec{p}_2, \quad \vec{p}_3=Ex_3R\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}, \quad \vec{p}_4=Ex_4R\cdot\begin{pmatrix}\sin\theta\cos\psi\\ \sin\theta\sin\psi\\ -\cos\theta\end{pmatrix}, \qquad (31)$$

where $0 \le E, x_3, x_4$ and

$$R = \begin{pmatrix} 1 & & \\ & \cos\chi & \sin\chi \\ & -\sin\chi & \cos\chi \end{pmatrix} \quad \text{and} \quad (\vec{p}_3 + \vec{p}_4 + \vec{p}_5) = 0. \tag{32}$$

Due to the ratoational symmetry of the spin-averaged matrix element in the xy-plane (around the axis of e^-–e^+ collision), it is enough to evaluate the matrix element only when $\vec{p}_3$ is purely in the yz-plane. Now, use the energy conservation to derive

$$0 \le x_3, x_4 \le 1, \quad 1 \le x_3 + x_4, \qquad \frac{1+\cos\theta}{2} = \frac{x_3 + x_4 - 1}{x_3 x_4}. \tag{33}$$

(e) Verify that $p_3 \cdot p_5 = 2E^2(1 - x_4)$ and $p_4 \cdot p_5 = 2E^2(1 - x_3)$.

(f) Verify that

$$\begin{aligned} d\sigma =& \frac{1}{(2E)^2 \times 2} \prod_{i=3}^{5} \left[\frac{d^3\vec{p}_i}{(2\pi)^3} \frac{1}{2E_{\vec{p}_i}} \right] (2\pi)^4 \delta^3(\sum_i \vec{p}_i) \delta(E_3 + E_2 + E_5 - 2E) \overline{|\mathcal{M}|^2}, \\ =& \frac{d\cos\chi}{(2\pi)^2} \frac{d\psi}{(2\pi)^2} \frac{dx_3 dx_4}{8^2} \overline{|\mathcal{M}|^2}. \end{aligned} \tag{34}$$

(g) Verify that

$$\begin{aligned} dx_3 dx_4 \int_{-1}^{1} d(\cos\chi) \int_0^{2\pi} d\psi \left\{ (p_1 \cdot p_3)^2 + (p_2 \cdot p_3)^2 + (p_1 \cdot p_4)^2 + (p_2 \cdot p_4)^2 \right\} \\ = dx_3 dx_4 \frac{32\pi}{3} E^4 (x_3^2 + x_4^2). \end{aligned} \tag{35}$$

Combining (34, 20, 35) and the problem (e) above, derive the following formula for the differential cross section of $e^- + e^+ \to \mu^- + \mu^+ + \gamma$:

$$\frac{d\sigma}{dx_3 dx_4} = \left[\frac{4\pi\alpha_e^2}{3s} \right] \times \frac{\alpha_g}{2\pi} (Q_{(\mu)} \widetilde{Q}_{(\mu)})^2 \frac{x_3^2 + x_4^2}{(1 - x_3)(1 - x_4)}, \tag{36}$$

where $\widetilde{Q}_{(\mu)} = Q_{(\mu)}$ and $\alpha_g = \alpha_e$ in this case (where γ is in the final state); a combination $4\pi\alpha_e/(3s)$ is the total cross section of the $e^+ + e^- \to \mu^+ + \mu^-$ process(?). The differential cross section for the process $e^- + e^+ \to q + \bar{q} + g$ is obtained by making a replacement (21):

$$\frac{d\sigma}{dx_3 dx_4} = \left[\frac{4\pi\alpha_e^2}{3s} \right] \times \frac{\alpha_g}{2\pi} Q_q^2 C_2 N_c \frac{x_3^2 + x_4^2}{(1 - x_3)(1 - x_4)}. \tag{37}$$

(h) One can see that the cross section for this $e^- + e^+ \to \mu^- + \mu^+ + V$ process is suppressed by a single power of $(\alpha_g/2\pi)$ relatively to the cross section of the 2-body final state; one can further see, however, that dx_3 and dx_4 integral has logarithmic divergence toward $x_3 \sim 1$ or / $x_4 \sim 1$, or simultaneously $x_3 \sim x_4 \sim 1$. To help understand what is happening in this divergence, figure out what kind of geometric configurations of $\vec{p}_{3,4,5}$ does the situation i) $x_4 \sim 1$, ii) $x_3 \sim 1$ and iii) both $x_{3,4} \sim 1$ correspond to. [We will not discuss how to make sense of this logarithmically divergent cross section in the QFT II course; if this logarithmic divergence is somehow replaced by a large logarithm, however, one will see that a possibly small factor $(\alpha_g/2\pi)$ can be compensated by the large logarithm, and the cross section of this 2 to 3 process can be comparable to the $e^- + e^+ \to \mu^- + \mu^+$ process.]