QFT II/QFT homework XI (Dec. 14, 2020)

- Reports on these homework problems are supposed to be submitted through the U Tokyo ITC-LMS. We request that the file name includes the problem number, such as II-1***.pdf or ****-IV-2-IX-1.jpeg. The ITC-LMS will show who had submitted the file (student ID and name), so the file name will not have to contain your name or ID number. (this instruction may be updated later)

1. **M1, E2 transitions etc.** [C]
   Electric dipole emission (E1 transition) is not the only possible mechanisms of transitions between atomic energy eigenstates. Explore more about those higher order effects, following your intellectual curiosity. References include ....

   - Landau Lifshitz vol 4 *Quantum Electrodynamics*, §45–50,
   - TAKAYANAGI, Kazuo *Genshi-bunshi Butsuri-gaku* (Asakura Publ. Co) written in Japanese, §4.4.2–4.4.5
   - look up online

2. **Positronium Decay** [B (or C)]
   Let us work out how to use Bethe–Salpeter wavefunction to compute the decay rate of a positronium (a bound state of a pair of e^-e^+) to two photons. Here, we need to note that each photon carries energy that is approximately m_e (in the rest frame of the initial bound state). The photon momenta, or derivatives acting on a photon field in the Lagrangian, is therefore not smaller than m_e. So, $\vec{\partial}/m_e$-expansion is not particularly useful in computing the matrix element for the decay rate. For this computation, it is better to use the frame of four-component spinor where

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & \\ & -\mathbf{1} \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} & \vec{\tau} \\ -\vec{\tau} & \end{pmatrix}. \tag{1}$$

   The Bethe–Salpeter wavefunction $\chi(p)$ in (here, $p^\mu := (p_1 - p_2)^\mu/2 =: (\omega, \vec{p})$)

$$\langle \Omega | T\left\{ \Psi(p_1)\overline{\Psi}(p_2) \right\} |s_{\rm tot}, s_z; n, l, m\rangle = (2\pi)^4 \delta^4(p_1 + p_2 - p_{\rm CM}) \left[\chi_{nls;s_z m}(p)\right]_{4\times 4} \tag{2}$$

is a 4×4 matrix valued function, and is approximately given by

$$[\chi_{nls;s_z m}(\omega, \vec{p})]_{4\times 4} \simeq \begin{pmatrix} \mathbf{1}_{2\times 2} \\ \frac{\vec{p}\cdot\vec{\tau}}{2m_e} \end{pmatrix} [P(s_{\rm tot}, s_z)]_{2\times 2} \left(-\frac{\vec{p}\cdot\vec{\tau}}{2m_e}, \; -\mathbf{1}_{2\times 2} \right) \chi_{nlm}(\omega, \vec{p}), \tag{3}$$

$$[P(s_{\rm tot}, s_z)]_{2\times 2} = \begin{cases} \mathbf{1}_{2\times 2} & s_{\rm tot} = 0 \\ \tau^3 & s_{\rm tot} = 1, \; s_z = 0, \\ (\tau^1 \pm i\tau^2)/\sqrt{2} & s_{\rm tot} = 1, \; s_z = \pm 1 \end{cases} \tag{4}$$

$$\int \frac{d\omega}{2\pi} \chi_{nlm}(\omega, \vec{p}) \simeq \sqrt{4m_e} \psi^{\rm NRQM}_{nlm}(\vec{p}), \tag{5}$$

using the Fourier transform of the wavefunction of a state $|n, l, m\rangle$ (with the reduceed mass $m_e/2$) in the non-relativistic quantum mechanics (that is, $\psi^{\rm NRQM}_{nlm}(\vec{p})$). In the rest of this problem, we set $\vec{p}_{\rm CM} = \vec{0}$.

(a) Verify that the matrix element of positronium $\to \gamma + \gamma$ is given by

$$i\mathcal{M} \simeq \int \frac{d\omega}{2\pi} \int \frac{d^3\vec{p}}{(2\pi)^3} (-ieQ_e)^2 \; \epsilon^*_\mu(\vec{k}) \epsilon^*_\nu(-\vec{k}) \tag{6}$$
$$\mathrm{tr}_{4\times 4} \left[\left(\frac{\gamma^\nu i[\omega\gamma^0 - (\vec{p} - \vec{k})^i \gamma^i + m_e]\gamma^\mu}{\omega^2 - (\vec{p} - \vec{k})^2 - m_e^2} + \frac{\gamma^\mu i[\omega\gamma^0 - (\vec{p} + \vec{k})^i \gamma^i + m_e]\gamma^\nu}{\omega^2 - (\vec{p} + \vec{k})^2 - m_e^2} \right) [\chi_{4\times 4}] \right],$$

if this expression is not trivial for you.

(b) Because $|\vec{k}| = m_e + (\Delta E)/2 \approx \mathcal{O}(m_e)$, while $\vec{p}$ is typically $\mathcal{O}(m_e\alpha)$ and ω even less, it makes sense to drop all of ω and $\vec{p}$ (and retain only m_e and $\vec{k}$) from the vertex-and-propagator $(\cdots + \cdots)$ part in the expression above. You will then notice that $d\omega$ integral can be carried out, and χ turns into $\psi^{\rm NRQM}$. Now, carry out the rest of the computation to find the decay rate of the positronium $(n, l, m, s_{\rm tot}, s_z) = (1, 0, 0; 0, 0)$ state. [It is not as important to get the O(1) coefficient precisely as to get the right power of m_e and α.] You will also be able to confirm that the two outgoing photons have oppsite angular momentum.

(c) Peskin–Schroeder Problem 5.4 (at the end of chapter 5) contains more information. If you are interested, you might think of exploring more. (category [C] then)

3. **Partial wave decomposition at work** [B]
Let us get the feeling how the partial wave decomposition works in practice, using the results of perturbative computations of 2body to 2body scattering amplitudes.

(a) We begin with the easiest example. Let us think of a 2-body to 2-body scattering in the s-channel, where a pari of scalar particles $\Phi^-(p_1)$ and $\Phi^+(p_2)$ coupled to a photon annihilates in pair and produce another pair of scalar particles $\Phi'^-(p_3)$ and $\Phi'^+(p_4)$. For simplicity, we only deal with the case where the center of mass energy is much higher than their rest mass (so that the mass parameters are negligible). The scattering amplitude is

$$\mathcal{M} = (-e^2 Q_\Phi Q_{\Phi'}) \frac{(p_1 - p_2) \cdot (p_3 - p_4)}{s} \simeq (-e^2 Q_\Phi Q_{\Phi'}) \frac{u - t}{2s} \simeq (-e^2 Q_\Phi Q_{\Phi'}) \frac{\cos\theta}{2}, \tag{7}$$

where θ is the scattering angle in the center of mass frame. Verify, by fitting the result above into the following expansion,

$$\frac{\mathcal{M}}{2(4\pi)^2} \simeq \mathcal{M}_{\rm red} = \sum_{\ell=0}^{\infty} Y_{\ell,m}(\hat{\mathbf{p}}_3) \left[\mathcal{M}_\ell(s)\right] \left(Y_{\ell,m}(\hat{\mathbf{p}}_1)\right)^{\rm cc}, \tag{8}$$

$$Y_{\ell,m=0}(\hat{n}) = P_\ell(\cos\theta) \sqrt{\frac{2\ell+1}{4\pi}}, \tag{9}$$

that only the $\ell = 1$ partial wave is non-zero in this scattering, and that

$$\mathcal{M}_{\ell=1}(s) \simeq \frac{-\alpha(Q_\Phi Q_{\Phi'})}{12}. \tag{10}$$

[So, in this example, $\mathcal{M}_{\ell=1}$ turns out to be independent of the center of mass energy $\sqrt{s}$, at this tree level calculation. The S-matrix in this $\ell = 1$ partial wave is $S_{\ell=1} \simeq 1 + i(-\alpha Q Q')/12 \simeq e^{-i\alpha QQ'/12}$, while $S_{\ell\neq 1} = 1$ in all other partial waves.]

(b) (If you are also interested in working on this...) Let us now consider a little more complicated case, where the initial state is not a pair of scalar $\Phi^- + \Phi^+$, but a pair of spin-1/2 fermions, $e^-(p_1) + e^+(p_2)$. We still consider the case where the final state is a pair of scalars $\Phi'^-(p_3) + \Phi'^+(p_4)$. We know that the scattering amplitude (at the center of mass frame, in the relativisitic limit, $\hat{\mathbf{p}}_1 = -\hat{\mathbf{p}}_2 = \hat{e}_z$) is given by

$$\mathcal{M} = (e^2 Q_e Q_{\Phi'}) \operatorname*{tr}_{2\times 2} \left[\begin{pmatrix} 0 & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & \end{pmatrix} \left(\xi_{e^-} \otimes \xi^\dagger_{e^+} \right) \right] \tag{11}$$

where θ and ϕ indicate the direction of the momentum $\hat{\mathbf{p}}_3$ after the scattering (in the center of mass frame). A 2×2 matrix

$$\xi_{e^-} \otimes \xi^\dagger_{e^+} = \begin{pmatrix} \frac{s_0^0 - s_0^1}{\sqrt{2}} & s_+^1 \\ -s_-^1 & \frac{s_0^0 + s_0^1}{\sqrt{2}} \end{pmatrix} \tag{12}$$

is a spin wavefunction; basis elements $|s_1^z, s_2^z\rangle$ correspond to

$$|1/2,1/2\rangle \to \xi\circ\xi^\dagger = \begin{pmatrix} 0 & 1 \\ & \end{pmatrix}, \quad |1/2,-1/2\rangle \to \xi\circ\xi^\dagger = \begin{pmatrix} -1 & 0 \\ & \end{pmatrix},$$
$$|-1/2,1/2\rangle \to \xi\circ\xi^\dagger = \begin{pmatrix} & \\ 0 & 1 \end{pmatrix}, \quad |-1/2,-1/2\rangle \to \xi\circ\xi^\dagger = \begin{pmatrix} & \\ -1 & 0 \end{pmatrix}.$$

We wish to consider the partial wave decomposition of this amplitude now.

Instead of the decomposition (8) for a 2-to-2 scattering between two spin-0 particles, we need to use

$$\begin{aligned}\mathcal{M}_{\rm red} = \sum_{j,j_z} \sum_{\ell,m,s3z,s4z} & C_{s3,s4,\ell}(j,j_z;s_{3z},s_{4z},m)\,(Y_{\ell,m}(\theta,\phi))\,[\mathcal{M}_j(s)] \\ & \sum_{\ell',m',s1z,s2z} C_{s1,s2,\ell'}(j,j_z;s_1^z,s_2^z,m')\,(Y_{\ell',m'}(\theta',\phi'))^{\rm cc} \\ & \qquad |s_3^z,s_4^z\rangle\langle s_1^z,s_2^z|, \end{aligned} \tag{13}$$

where $C_{s1,s2,\ell}(j,j_z;s_{1z},s_{2z},m)$ is the Clebsch–Gordan coefficient describing the irreducible decomposition $(\text{spin}s_1)\otimes(\text{spin}s_2)\otimes(\text{spin}\ell) \simeq \cdots\oplus(\text{spin}j)\oplus\cdots$. The partial wave amplitude $[\mathcal{M}_j]$ for a given total angular momentum j is not just a complex number (for a given center of mass energy $\sqrt{s}$) but a matrix, because there may be multiple ways to add spins s_1, s_2 [resp. s_3, s_4] and the angular momentum ℓ' [resp. ℓ] of some relative wavefunction of $\hat{\mathbf{p}}_1$ [resp. $\hat{\mathbf{p}}_3$] to obtain j.

i. (this is not a problem) In this part (b), we still consider the case the final state particles Φ'^- and Φ'^+ are spin-0 particles ($s_3 = s_4 = 0$), so we only need to use $C_{0,0,\ell}(j,j_z;0,0,m) = \delta_{\ell,j}\delta_{j_z,m}$. Looking at the (θ,ϕ) dependence of the amplitude (11) and using the spehreical harmonics

$$Y_{\ell=1}^{m=\pm1} = \sqrt{\frac{3}{4\pi}}\frac{\mp1}{\sqrt{2}}\sin\theta e^{\pm\phi}, \qquad Y_{\ell=1}^{m=0} = \sqrt{\frac{3}{4\pi}}\cos\theta, \tag{14}$$

we find that only the $j=\ell=1$ term should be retained in the expansion (13).

$[\mathcal{M}_{j=1}]$ is a 1×4 matrix. The "$1\times$" part must already be obvious. The "$\times4$" part is also understood as follows. For the initial state, $s_1 = s_2 = 1/2$. So, $s_{\rm tot} = 1$ or 0. For $s_{\rm tot}\otimes\ell'$ to contain $j=1$, the only possibilities are $(s_{\rm tot},\ell') = (0,1),(1,0),(1,1),(1,2)$. The 1×4 entries of the matrix $[\mathcal{M}_{j=1}]$ are denoted by $([\mathcal{M}_1^{(0,1)}],[\mathcal{M}_1^{(1,0)}],[\mathcal{M}_1^{(1,1)}],[\mathcal{M}_1^{(1,2)}])$, using $(s_{\rm tot},\ell')$ as the label.

ii. (still this is not a problem) The scattering amplitude is

$$\mathcal{M}_{\rm red} = \frac{\alpha Q_\Phi Q_{\Phi'}}{8\pi}\left(Y_1^1(\theta,\phi)\sqrt{\frac{8\pi}{3}}(-s_+^1) + Y_1^{-1}(\theta,\phi)\sqrt{\frac{8\pi}{3}}(-s_-^1)\right), \tag{15}$$

or equivalently,

$$\mathcal{M}_{\rm red} = \frac{\alpha Q_\Phi Q_{\Phi'}}{\sqrt{24\pi}}\left(Y_1^1(\theta,\phi)\begin{pmatrix} & \\ -1 & 0\end{pmatrix} + Y_1^{-1}(\theta,\phi)\begin{pmatrix} 0 & 1 \\ & \end{pmatrix}\right), \tag{16}$$

$$= -\frac{\alpha Q_\Phi Q_{\Phi'}}{\sqrt{24\pi}}\left(Y_1^1(\theta,\phi)\langle 1/2, 1/2| + Y_1^{-1}(\theta,\phi)\langle -1/2,-1/2|\right). \tag{17}$$

The operator form $\mathcal{M}_{\rm red}^{\rm op}$ in (16, 17) becomes the amplitude $\mathcal{M}_{\rm red}^{\rm amp}$ in (15) when we evaluate the former on the spin wavefunction (12); ${\rm tr}_{2\times 2}[\mathcal{M}^{\rm op}_{\rm red}(\xi\circ\xi^\dagger)] = \mathcal{M}^{\rm amp}_{\rm red}$. The following translation is understood in the equality between (16) and (17).

$$\langle 1/2, 1/2| \Leftrightarrow \begin{pmatrix} & \\ 1 & 0\end{pmatrix}, \quad \langle 1/2, -1/2| \Leftrightarrow \begin{pmatrix} -1 & 0 \\ & \end{pmatrix},$$
$$\langle -1/2, 1/2| \Leftrightarrow \begin{pmatrix} & \\ 0 & 1\end{pmatrix}, \quad \langle -1/2, -1/2| \Leftrightarrow \begin{pmatrix} 0 & -1 \\ & \end{pmatrix}.$$

iii. Now, complete the partial wave decomposition. That can be done by setting

$$\frac{\alpha Q_\Phi Q_{\Phi'}}{\sqrt{24\pi}}(-s_+^1) \tag{18}$$
$$= [\mathcal{M}_1^{(0,1)}]s_0^0\times 0 + [\mathcal{M}_1^{(1,0)}][Y_0^0]_0 s_+^1 + [\mathcal{M}_1^{(1,1)}][Y_1^0]_0\frac{s_+^1}{\sqrt{2}} + [\mathcal{M}_1^{(1,2)}][Y_2^0]_0\frac{s_+^1}{\sqrt{10}},$$

$$0 \tag{19}$$
$$= [\mathcal{M}_1^{(0,1)}][Y_1^0]_0 s_0^0 + [\mathcal{M}_1^{(1,0)}][Y_0^0]_0 s_0^1 + [\mathcal{M}_1^{(1,1)}][Y_1^0]_0\times 0 + [\mathcal{M}_1^{(1,2)}][Y_2^0]_0\frac{-2s_0^1}{\sqrt{10}},$$

and

$$\frac{\alpha Q_\Phi Q_{\Phi'}}{\sqrt{24\pi}}(-s_-^1) \tag{20}$$
$$= [\mathcal{M}_1^{(0,1)}]s_0^0\times 0 + [\mathcal{M}_1^{(1,0)}][Y_0^0]_0 s_-^1 + [\mathcal{M}_1^{(1,1)}][Y_1^0]_0\frac{-s_-^1}{\sqrt{2}} + [\mathcal{M}_1^{(1,2)}][Y_2^0]_0\frac{s_-^1}{\sqrt{10}},$$

for $j_z = +1, 0$ and -1, respectively. Here, in writing down the right hand sides, we used the Clebsch–Gordan coefficients relating $|j=1, j_z\rangle^{(s_{\rm tot},\ell')}$ and $|s_{\rm tot}; s^z_{\rm tot}\rangle|\ell'; m'\rangle$:

$$
\begin{aligned}
(s_{\rm tot}\otimes\ell') \supset (j=1) \ni |1,+\rangle^{(1,1)} &= \frac{|1;+\rangle|1;0\rangle - |1;0\rangle|1;+\rangle}{\sqrt{2}}, \\
|1,0\rangle^{(1,1)} &= \frac{|1;+\rangle|1;-\rangle - |1;-\rangle|1;+\rangle}{\sqrt{2}}, \\
|1,-\rangle^{(1,1)} &= \frac{|1;0\rangle|1;-\rangle - |1;-\rangle|1;0\rangle}{\sqrt{2}},
\end{aligned}
$$

and

$$
\begin{aligned}
|1,+\rangle^{(1,2)} &= \frac{|1;+\rangle|2;0\rangle - \sqrt{3}|1;0\rangle|2;1\rangle + \sqrt{6}|1;-\rangle|2;2\rangle}{\sqrt{10}}, \\
|1,0\rangle^{(1,2)} &= \frac{\sqrt{3}|1;+\rangle|2;-1\rangle - 2|1;0\rangle|2;0\rangle + \sqrt{3}|1;-\rangle|2;1\rangle}{\sqrt{10}}, \\
|1,-\rangle^{(1,2)} &= \frac{\sqrt{6}|1;+\rangle|2;-2\rangle - \sqrt{3}|1;0\rangle|2;-1\rangle + |1;-\rangle|2;0\rangle}{\sqrt{10}},
\end{aligned}
$$

and also used the fact that

$$
Y^{m'}_{\ell'}(\cos\theta = 1, {}^\forall\phi) = \delta_{m'}\sqrt{\frac{2\ell'+1}{4\pi}} =: [Y^0_{\ell'}]_0\delta_{m'}. \tag{21}
$$