QFT II/QFT homework XII (Dec. 21, 2020)

- Reports on these homework problems are supposed to be submitted through the U Tokyo ITC-LMS. We request that the file name includes the problem number, such as II-1***.pdf or ****-IV-2-IX-1.jpeg. The ITC-LMS will show who had submitted the file (student ID and name), so the file name will not have to contain your name or ID number. (this instruction may be updated later)

1. **Neutrino Mass and Partial Wave Unitarity** [A]
   There are two different ways so neutrinos have non-zero masses. One of the two ways is the Majorana scenario, where

$$\mathcal{L} \supset \bar{\psi}_{\dot{\alpha}} i \bar{\sigma}^{\dot{\alpha}\alpha}_{\mu} D^{\mu} \psi_{\alpha} + \mathcal{L}_{\rm dim5}, \qquad \mathcal{L}_{\rm dim5} = \frac{1}{2M} \epsilon^{\alpha\beta} (\psi_{\alpha}\phi_0)(\psi_{\beta}\phi_0) + {\rm h.c.}. \tag{1}$$

   ψ is a 2-component Grassmann field (see lecture note Week 03 for more details) with the two components labeled by $\alpha = 1, 2$. ϕ_0 is a complex scalar field. $\epsilon^{\alpha\beta}$ is the totally anti-symmetric tensor with $\epsilon^{12} = +1$. When the field ϕ_0 develops non-zero vacuum expectation value, the dimension-5 operator in $\mathcal{L}_{\rm dim5}$ contains a term that is bilinear in the fluctuation (ψ), which is regarded as a mass term of the fluctuation.

   (a) Verify that the scattering amplitude of $\nu(\vec{p}_1) + \nu(\vec{p}_2) \to \bar{\phi}_0(\vec{p}_3) + \bar{\phi}_0(\vec{p}_4)$ is

$$i\mathcal{M}^{\rm LO} = i\frac{\sqrt{2E_1 2E_2}}{M} = i\frac{\sqrt{s}}{M} \tag{2}$$

   at the tree level leading order. Use the relativistic approximation, and the center of mass frame.

   (b) Verify that only the $j = 0$ partial wave is present in this scattering amplitude, and that

$$\begin{aligned} &Y^0_0(\theta,\phi)\ [\mathcal{M}_{j=0}]^{\rm LO} C_{(1/2,1/2,\ell')}(0,0;s^z_1,s^z_2,m')([Y^{m'}_{\ell'}]_0)^{\rm c.c.} \langle s^z_1, s^z_2| \\ &\quad = Y^0_0(\theta,\phi)\frac{1}{(8\pi)}\frac{\sqrt{s}}{M}\left((Y^0_0)^{\rm cc}_0 \frac{(\langle\downarrow\uparrow| - \langle\uparrow\downarrow|)}{\sqrt{2}} + \frac{1}{\sqrt{3}}(Y^0_1)^{\rm cc}_0\frac{(\langle\downarrow\uparrow| + \langle\uparrow\downarrow|)}{\sqrt{2}}\right). \end{aligned} \tag{3}$$

   Here, $(Y^{m'=0}_{\ell'})_0 := Y^{m'}_{\ell'}(\theta'=0,\phi') = \delta_{m'}\sqrt{(2\ell'+1)/(4\pi)}$.

   (c) According to the tree level leading order result, the $j = 0$ partial wave $1 + i[\mathcal{M}_{j=0}]^{\rm LO}$ has an imaginary part larger than $\mathcal{O}(1)$ for large $\sqrt{s}$. The partial wave unitarity will be restored, presumably due to presence of extra contributions to $[\mathcal{M}_{j=0}]$ that are at least just as large as $[\mathcal{M}_{j=0}]^{\rm LO}$ at energy scale

$$\sqrt{s} \gtrsim 8\pi M. \tag{4}$$

That is an indication that the Standard Model with $\mathcal{L}_{\rm dim5}$ is not going to be a good approximation of the interactions of elementary particles at least at such high energy scale.

To account for the atmospheric neutrino oscillation in the Majorana scenario, we need

$$\frac{(\langle\phi_0\rangle)^2}{M} \gtrsim \sqrt{|\Delta m^2_{\rm atm}|} \simeq \sqrt{2\times 10^{-3}}\ {\rm eV}, \tag{5}$$

where $\langle\phi_0\rangle \simeq 174$GeV is the Higgs boson vacuumm expectation value. So, extra contributions can be negligible only in the energy range

$$\sqrt{s} \lesssim 8\pi M \lesssim 8\pi \frac{(\langle\phi_0\rangle)^2}{\sqrt{|\Delta m^2_{\rm atm}|}}. \tag{6}$$

Work out the upperbound on the right-hand side.

2. **Froissart–Martin bound** [C]
   not enough time to prepare this problem this year.

3. **strongly coupled dark matter**
   not enough time to prepare this problem this year.