QFT II/QFT homework XIV (Jan. 18, 2021)

- Reports on these homework problems are supposed to be submitted through the U Tokyo ITC-LMS. We request that the file name includes the problem number, such as II-1***.pdf or ****-IV-2-IX-1.jpeg. The ITC-LMS will show who had submitted the file (student ID and name), so the file name will not have to contain your name or ID number. (this instruction may be updated later)

1. **Imaginary time tau-ordered propagator: off-shell form** [B]
Presumably at the end of Week 14 lecture, we have derived the imaginary time tau-ordered propagator

$$\langle T_{au} \left\{ \phi(\vec{x}+\vec{y}, \tau+\tau_0)\phi(\vec{y}, \tau_0) \right\} \rangle_\beta, \qquad 0 \leq \tau_0, \tau_0+\tau \leq \beta \tag{1}$$

in a free real scalar field theory on $(d+1)$-dimensional space-time:

$$D(\vec{x},\tau)|_{0<\tau<\beta} = \int \frac{d^dk}{(2\pi)^d}\frac{1}{2E_k}\left(e^{i\vec{k}\cdot\vec{x}-E_k\tau}\frac{1}{1-e^{-E_k\beta}} + e^{-i\vec{k}\cdot\vec{x}+E_k\tau}\frac{e^{-E_k\beta}}{1-e^{-E_k\beta}}\right). \tag{2}$$

An expression for $\tau < 0$ is omitted here. Now, let us think of expressing this propagator in the full Fourier expansion, not just in the d space directions but also in the imaginary time direction. That is to introduce $\tilde{D}$ that fits into

$$D(\vec{x},\tau)|_{0<\tau<\beta} =: T\sum_{m\in\mathbb{Z}}\int \frac{d^3k}{(2\pi)^d} e^{i\vec{k}\cdot\vec{x}} e^{i\tau(2\pi/\beta)m}\tilde{D}(\vec{k},(2\pi/\beta)m). \tag{3}$$

Verify that

$$\tilde{D}(\vec{k},(2\pi/\beta)m) = \frac{1}{((2\pi/\beta)m)^2+E_k^2}; \tag{4}$$

Tihs concise result may well be regarded as analytic contiunation of the Minkowski space propagator $k^0 \to i(2\pi/\beta)m$. Also, this expression is intuitively acceptable, because the partition function of a free scalar field theory with imaginary time is the Gaussian integral with the exponent proportional to $((2\pi/\beta)m)^2 + \vec{k}^2 + m^2$.

2. **Path ingegral derivation of the free energy formula** [A-XIV-f]
In the Week 14 lecture, the free energy formula of a harmonic oscillator

$$e^{-\beta F} = Z_{\text{harm.oscl.}}[\beta,\omega] = \text{tr}\left[e^{-\beta H}\right] = e^{-\beta\omega/2}\frac{1}{1-e^{-\beta\omega}} = \frac{1}{e^{\beta\omega/2}-e^{-\beta\omega/2}} \tag{5}$$

is derived by using the operator formalism. It must be possible also to derive the same result using the path integral formulation of the same system, so let us try.

In the Week 1 lecture, the time evolution kernel

$$e^{-iHT} = \int dq_f \int dq_0 \, |q_f\rangle\langle q_i| \;\; \langle q_f|e^{-iHT}|q_0\rangle \tag{6}$$

between the time t_0 and $t_0 + T$ was worked out:

$$\langle q_f|e^{-iHT}|q_0\rangle = \sqrt{\frac{m\omega}{2\pi i \sin(\omega T)}} e^{iS_{\rm cl}}, \tag{7}$$

$$S_{\rm cl} = \frac{m\omega}{2}\left(\frac{q_f^2 + q_0^2}{\tan(\omega T)} - \frac{2q_0 q_f}{\sin(\omega T)}\right). \tag{8}$$

Now, by substituting $T \to -i\beta$ and taking trace of the operator (6), confirm that the same result is reproduced.