QFT II/QFT homework XV (Jan. 25)

- Reports on these homework problems are supposed to be submitted through the U Tokyo ITC-LMS. We request that the file name includes the problem number, such as II-1***.pdf or ****-IV-2-IX-1.jpeg. The ITC-LMS will show who had submitted the file (student ID and name), so the file name will not have to contain your name or ID number. (this instruction may be updated later)

1. **Non-rela effective theories via path integration** [B]

   (a) We have dealt with the process of deriving the effective theory of non-relativisitic two component fermion from Dirac Lagrangian in a couple of different perspectives so far. This homework problem provides one more take on this phenomenon. Let us use the gamma matrices of the form

$$\gamma^0 = \begin{pmatrix} \mathbf{1}_{2\times 2} & \\ & -\mathbf{1}_{2\times 2} \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} & \vec{\tau} \\ -\vec{\tau} & \end{pmatrix}, \tag{1}$$

   and the four component Dirac fermion be split into the upper two and lower two components in this frame,

$$\Psi =: e^{-iMt} \begin{pmatrix} \psi \\ \bar{\chi} \end{pmatrix}. \tag{2}$$

   i. Now, rewrite the Dirac Lagrangian

$$\mathcal{L}_{\rm Dirac} = \Psi^\dagger \gamma^0 \left(i\gamma^\mu(\partial_\mu + ieQ_e A_\mu) - M \right) \Psi \tag{3}$$

   in terms of ψ and χ. [You will find that the mass parameter M cancels in the coefficient of $\psi^\dagger\psi$, while it does not in the coefficient of $\chi\chi^\dagger$.]

   ii. Complete a square with respect to χ–$\chi^\dagger$, and carry out the Gaussian integral with respect to

$$\mathcal{D}\chi^\dagger \mathcal{D}\chi \subset \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}\bar{\chi}\mathcal{D}\chi = \mathcal{D}\Psi\mathcal{D}\bar{\Psi}. \tag{4}$$

   [The remnant of completion of a square in $\mathcal{L}_{\rm Dirac}$ must be the effective theory Lagrangian of the non-relativisitic two component fermion.]

(b) If one hopes to write down a quantum field theory for a spin-0 particle whose number conserves (imagine an alkali atom), the field theory of a complex scalar field will usually be the first choice. If you are interested only in using it in a non-relativisitc regime, however, it is not necessary to retain both the particle and its anti-particle in your theory; there must be a low-energy effective theory where you only retain the particle without its anti-particle. What is the process of deriving this low-energy effective theory like? Here is how.

i. The path integral of a theory of a complex scalar field Φ is carried out over the space with the coordinates $(\Phi(k^0,\vec{k}), \Phi^*(k^0,\vec{k}))$. Let $\phi_+(k^0,\vec{k})$ and $\phi_-(k^0,\vec{k})$ for $\omega > 0$ be the positive and negative frequency parts of $\Phi(k^0,\vec{k})$; similarly, we put the positive and negative frequency parts of Φ^* as ϕ_-^* and ϕ_+^*, respectively. The path integral measure is now

$$\mathcal{D}\phi_+\mathcal{D}\phi_-\mathcal{D}\phi_-^*\mathcal{D}\phi_+^*. \tag{5}$$

When you start from

$$\mathcal{L}_{\mathrm{KG}} = (\partial_\mu\Phi)^*(\partial^\nu\Phi) - M^2|\Phi|^2, \tag{6}$$

verify that this Lagrangian is already in the form of a sum of a square (so we do not need to complete a square), and that we are ready to integrate out ϕ_- and ϕ_-^*.

ii. (this is a remark, not a problem) Now you are left with $\mathcal{D}\phi_+(k^0_{\geq 0},\vec{k})$ and $\mathcal{D}\phi_+^*(k^0_{\leq 0},\vec{k})$. Quantization of this field theory leads to ϕ_+ containing just the annihilation operators of a particle, without the creation opertor of its anti-particle, because the $\phi_+(k^0_{\geq 0},\vec{k})$ field does not admit a negative frequency solution to the equation of motion.

iii. Rewrite the effective theory Lagrangian of ϕ_+–ϕ_+^* in terms of a new pair of fields $\phi_+ =: e^{-iMt}\underline{\phi}_+$ and $\phi_+^* =: e^{+iMt}\underline{\phi}_+^*$. Once you have done that, you will presumably feel motivated to make a furtuer redefinition, $\underline{\phi}_+ := \underline{\underline{\phi}}_+/\sqrt{2M}$. Verify then that the effective theory Lagrangian written in terms of $\underline{\underline{\phi}}_+$ and its Hermitian conjugate contains terms that look like the action for the Schroedinger equation.

iv. Suppose that we are interested in using this effective theory only in circumstances where $|\vec{k}| \ll M$ (that is, in non-relativistic situations). This is translated into the presence of a small parameter $\lambda \ll 1$ so that $|\vec{k}| \sim \lambda \times \mathcal{O}(M)$.

Then the operator $\vec{\partial}^2/M$ is given a scaling behavior $M \times \lambda^2$. This means that we should assign the same scaling behavior $M \times \lambda^2$ to the operator ∂_t. What is the scaling dimension of the operator $(\partial_t)^2/M$, then? [That fact that this operator is assigned a higher scaling dimension (simply the power of λ) than the two others justifies to drop this $(\partial_t)^2/M$ operator from consideration (or to treat this operator as a correction term).]

v. If you are interested, repeat the same procedure for a complex sclar field theory, but now with an interaction term $-\frac{\kappa}{4}|\Phi|^4$ term added to $\mathcal{L}_{\rm KG}$.

2. **Real-time formalism propagators and Fluctuation Dissipation theorem** [C]

(a) Consider a harmonic oscillator, where the unit excitation energy (frequency) is E. $\phi(t) = (ae^{-iEt} + a^\dagger e^{iEt})/\sqrt{2E}$, $p(t) = \sqrt{E/2}(ae^{-iEt} - a^\dagger e^{+iEt})/i$. Now, compute

$$\Delta^<(t) := \frac{\mathrm{Tr}\left[e^{-\beta H_0}\phi(0)\phi(t)\right]}{\mathrm{Tr}\left[e^{-\beta H_0}\right]}, \qquad \Delta^>(t) := \frac{\mathrm{Tr}\left[e^{-\beta H_0}\phi(t)\phi(0)\right]}{\mathrm{Tr}\left[e^{-\beta H_0}\right]}, \tag{7}$$

and find their expressions that use the Bose–Einstein distribution

$$n_E := \frac{1}{(e^{\beta E} - 1)}. \tag{8}$$

(b) Note that the τ-ordered propagator in the imaginary time formalism corresponds to

$$\begin{cases} \Delta^<(t \to -i\tau), & \text{if } \tau < 0, \\ \Delta^>(t \to -i\tau), & \text{if } \tau > 0. \end{cases} \tag{9}$$

(c) Verify, by using $e^{\beta H_0}\phi(t)e^{-\beta H_0} = \phi(t - i\beta)$, that $\Delta^<(t) = \Delta^>(t - i\beta)$.

(d) Now, we examine relations among those propagators in their Fourier-transformed version. As a preparation, verify that

$$\Theta(t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \frac{i}{\omega + i\epsilon}, \qquad -\Theta(-t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \frac{i}{\omega - i\epsilon}. \tag{10}$$

(e) Let

$$\langle 0| [\phi(t), \phi(0)] |0\rangle =: \int \frac{d\omega}{2\pi} e^{-i\omega t} \rho(\omega), \tag{11}$$

$$\Delta^R(t) = \Theta(t)\langle 0| [\phi(t), \phi(0)] |0\rangle =: \int \frac{d\omega}{2\pi} e^{-i\omega t} \widetilde{\Delta}^R(\omega), \tag{12}$$

$$\Delta^A(t) = -\Theta(-t)\langle 0| [\phi(t), \phi(0)] |0\rangle =: \int \frac{d\omega}{2\pi} e^{-i\omega t} \widetilde{\Delta}^A(\omega). \tag{13}$$

Verify that

$$\widetilde{\Delta}^R(\omega) = \int \frac{d\omega'}{2\pi} \frac{i}{\omega - \omega' + i\epsilon} \rho(\omega'), \qquad \widetilde{\Delta}^A(\omega) = \int \frac{d\omega'}{2\pi} \frac{i}{\omega - \omega' - i\epsilon} \rho(\omega'), \quad (14)$$

$$\mathrm{Re}\left(\widetilde{\Delta}^R(\omega)\right) = \frac{1}{2}\rho(\omega). \quad (15)$$

An example: in the case of a harmonic oscillator, $\langle 0| [\phi(t), \phi(0)] |0\rangle = (e^{-iEt} - e^{iEt})/(2E)$, so $\rho(\omega)$ is the following:

$$\rho(\omega) = \frac{2\pi}{2E} \left(\delta(\omega - E) - \delta(\omega + E)\right), \quad (16)$$

(f) Using the fact that $\Delta^R(t) = \Theta(t)(\Delta^>(t) - \Delta^<(t))$, and the fact that $\Delta^<(t) = \Delta^>(t - i\beta)$, derive the following relations on the Fourier transforms of $\Delta^<$ and $\Delta^<$:

$$\widetilde{\Delta}^<(\omega) = e^{-\beta\omega} \widetilde{\Delta}^>(\omega), \quad (17)$$

$$\rho(\omega) = \widetilde{\Delta}^>(\omega) - \widetilde{\Delta}^<(\omega) = (1 - e^{-\beta\omega})\widetilde{\Delta}^>(\omega). \quad (18)$$

(g) (remark, not a homework problem) Combining all the results we have derived above, we see that

$$\mathrm{Re}\left(\widetilde{\Delta}^R(\omega)\right) = \frac{1}{2}\rho(\omega) = \frac{1 - e^{-\beta\omega}}{1 + e^{-\beta\omega}} \frac{\left(\widetilde{\Delta}^> + \widetilde{\Delta}^<\right)}{2} = \tanh(\beta\omega/2) \frac{\left(\widetilde{\Delta}^> + \widetilde{\Delta}^<\right)}{2}. \quad (19)$$

This relation is an example of the Fluctuation–dissipation theorem; in fact, this relation holds not just for fields ϕ that are used for perturbative computations in a quantum field theory system, but also for any kinds of operators $\mathcal{O}$. Two point functions $\Delta^<(t)$ and $\Delta^>(t)$ are defined as in (7) by simply replacing ϕ by $\mathcal{O}$. The statement $\Delta^<(t) = \Delta^>(t - i\beta)$ still holds true. In the expressions (11–13), $[\phi(t), \phi(0)]$ is replaced by $[\mathcal{O}(t), \mathcal{O}(0)]$, and the vacuum expectation value by thermal average, because $[\mathcal{O}(t), \mathcal{O}(0)]$ is an operator in general, rather than a $\mathbb{C}$-number. The algebra that we have gone through, (10, 14–18), still holds true, and hence the relation (19) follows. The combination $(\widetilde{\Delta}^> + \widetilde{\Delta}^<)(\omega)$ in the right-hand side is the power spectrum (Fourier transform) of the thermal average of the fluctuation of the operator at the quadratic order, $\{\mathcal{O}(t), \mathcal{O}(0)\}$. On the other hand, $\widetilde{\Delta}^R$ on the left-hand side determines (linear) response of the system under an external time-varying field coupled to the operator $\mathcal{O}$; its real part[1] corresponds to the dissipation in the $\mathcal{O}$–$\mathcal{O}$ channel

[1]The imaginary part of the response function is not dissipative in nature. Imagine a free electron moving in an AC electric field. The current due to the oscillating electron motion tracks the oscillating electric field, but with the delay in phase by $\pi/2$, so the response function is pure imaginary.