Setting the initial state to the ground state (vacuum) or to the thermal distribution.

- $\langle q_f \text{ at } t_f | q_i \text{ at } t_i \rangle = \sum_n \psi_n(q_f)\, \psi_n^*(q_i)\, e^{-iE_n(t_f-t_i)}$.

$$[t_i \to -\infty] \Longrightarrow t_i \to -\infty(1-i\varepsilon) \qquad (i\varepsilon \text{ prescription})$$

$$[t_f \to +\infty] \Longrightarrow t_f \to +\infty(1-i\varepsilon).$$

$$\times\, e^{-i[(\Delta E)>0][(-i\varepsilon)\infty]} = e^{-\infty\cdot\varepsilon\cdot(\Delta E)>0}$$

excited state contributions die out.

As a result.

$$\boxed{\begin{aligned}&\int_{i\varepsilon\text{-prescription}} \mathcal{D}q\,\mathcal{D}p \; e^{i\int dt(p\dot{q}-H)} \; \prod_i f_i(q_i) \; \prod_j \tilde{f}_j(p_j)\\ &\propto \langle 0.\,\text{future} |\, T\Big\{ \prod_i f_i(q(t_i)) \; \prod_j \tilde{f}_j(p(t_j))\Big\} | 0.\,\text{past}\rangle\end{aligned}}$$

Heisenberg picture states

operators in the Heisenberg picture.

- Boltzmann weight.

$$Z = \sum_n e^{-\beta E_n} = \int dq \sum_n \psi_n(q)\, \psi_n^*(q)\, e^{-\beta E_n}$$

$$= \int d(q_i = q_f)\; \langle q_f \text{ at } (t_i - i\beta) | q_i \text{ at } t_i \rangle.$$

$$= \int_{q(t=-i\beta)=q(t=0)} \mathcal{D}q(\tau)\,\mathcal{D}p(\tau)\; e^{i\sum_{k=1}^{N}\{p_k(q_k-q_{k-1}) - (\Delta t = -i\Delta\tau) H(p_k, q_{k-1})\}}$$

$$= \int_{q(\tau=\beta)=q(\tau=0)} \mathcal{D}q(\tau)\mathcal{D}p(\tau)\; e^{\int d\tau(ip q' - H)}$$

$$\boxed{\begin{aligned}&\frac{\partial}{\partial\tau} \Rightarrow {}'\\ &t = -i\tau\end{aligned}}$$

If $H = \frac{p^2}{2m} + V(q)$ then

$$Z \propto \int_{q(\tau=\beta)=q(\tau=0)} dq \quad e^{-\int d\tau \left\{ \frac{m}{2} (q')^2 + V(q) \right\}}$$

after Gaussian integral.

The exponent:

instead of $i \int dt\, L = i \int dt \left\{ \frac{m}{2} (\dot{q})^2 - V(q) \right\}$

now $-\int d\tau \left\{ \frac{m}{2} (q')^2 + V(q) \right\}$ negative definite.

§1.2 Path integral formulation for a 2-state system.

Consider a quantum mechanical system with only 2 states.

$$\text{States} \iff |0\rangle c_0 + |1\rangle c_1 \qquad (c_0, c_1) \in \mathbb{C}^2$$

$$\{b, b^\dagger\} = 1 \qquad |1\rangle = b^\dagger |0\rangle \qquad b|0\rangle = 0$$

$$|0\rangle = b|1\rangle \qquad b^\dagger |1\rangle = 0$$

e.g. • an isolated atom with one e^- energy level
(2 states $\iff S_z = \pm 1/2$)

• one fermionic state without $S_z = \pm 1/2$ degeneracy (e.g. $\langle \vec{B} \rangle \neq 0$)
(2 states $\iff$ occupied or not)

Math preparation

Introduce a notion "Grassmann variables"

Instead of $q_i q_j = q_j q_i$, $\theta_i \theta_j = -\theta_j \theta_i$.

$$\Rightarrow \theta_i \theta_i = -\theta_i \theta_i = 0$$

"functions of Grassmann variables"

$$f(q's, \theta's) := f^{(0)}(q's) + \sum_i f_i^{(1)}(q's)\theta_i + \sum_{i<j} f_{ij}^{(2)}(q's)\theta_i\theta_j + \cdots$$
$$+ f_{\text{all}}^{(n)}(q's)(\theta_1\theta_2 \cdots \theta_n)$$

"derivative w.r.p.t. θ_i" $\quad f(q, \theta, \theta') = f^{(0)}(q) + f_1^{(1)}(q)\theta + f_2^{(1)}(q)\theta' + f_{12}^{(2)}(q)\theta\theta'$

$$\Rightarrow \left(\partial f / \partial \theta\right) = f_1^{(1)}(q) - f_{12}^{(2)}(q)\theta' \qquad \left(\partial\theta \backslash \partial f\right) = f_1^{(1)}(q) + f_{12}^{(2)}(q)\theta'$$

((pull out θ to the right)
use $\theta\theta' = -\theta'\theta$) $\qquad$ (pull out θ to the left)

"integrate w.r.p.t. θ_i"

$$\int d\theta \; f(q, \theta) = \int d\theta \left(f^{(0)}(q) + f^{(1)}(q)\theta \right) := f^{(1)}(q)$$

$$\int d\theta_2 d\theta_1 \left(f^{(0)}(q) + f_1^{(1)}(q)\theta_1 + f_2^{(1)}(q)\theta_2 + f_{12}^{(2)}(q)\theta_1\theta_2 \right)$$

$$= \int d\theta_2 \left(f_1^{(1)}(q) + f_{12}^{(2)}(q)\theta_2 \right) = f_{12}^{(2)}(q)$$

set $d\theta_1 d\theta_2 = -d\theta_2 d\theta_1$ so that the same result $f_{12}^{(2)}(q)$ is obtained when doing the $d\theta_2$ integration first.

The wavefunction of a state $|\text{state}\rangle = |0\rangle c_0 + |1\rangle c_1$ $(c_{0,1} \in \mathbb{C})$ can be expressed as $\Psi_{\text{state}}(\bar\theta) = c_0 + c_1\bar\theta$ by using a Grassmann variable $\bar\theta$.

Analogy of $\Psi_{\text{state}}(q) = \langle q|\text{state}\rangle$ in a bosonic (ordinary) QM system,

$$\boxed{\Psi_{\text{state}}(\bar\theta) = \left(\langle 0| + \bar\theta\langle 1|\right)|\text{state}\rangle}$$

Analogy of $|\text{state}\rangle = \int dq\, |q\rangle \Psi_{\text{state}}(q)$,

$$\boxed{|\text{state}\rangle = \int d\bar\theta d\theta \left(|0\rangle + |1\rangle\theta \right) e^{-\bar\theta\theta} \Psi_{\text{state}}(\bar\theta)}$$

verify by yourself that the RHS is $|0\rangle c_0 + |1\rangle c_1$.

Insertion of a complete system of states:

$$\boxed{\mathbb{1}_{2\times 2} = \int d\bar\theta d\theta \left(|0\rangle + |1\rangle\theta \right) e^{-\bar\theta\theta} \left(\langle 0| + \bar\theta\langle 1| \right)}$$

analogous to $\mathbb{1} = \int dq\, |q\rangle\langle q|$

The inner product

$$\langle \text{state}'|\text{state}\rangle = (c_0')^* c_0 + (c_1')^* c_1 \in \mathbb{C}$$

$$= \int d\bar\theta d\theta\, (c_0'^* + c_1'^*\theta)(c_0 + c_1\bar\theta) e^{-\bar\theta\theta} =: \int d\bar\theta d\theta\, \Psi^*_{\text{state}'}(\theta) \Psi_{\text{state}}(\bar\theta) e^{-\bar\theta\theta}$$

$$= \int d\bar\theta d\theta\, ((c_0')^* + (c_1')^*\theta) e^{-\bar\theta\theta} (c_0 + c_1\bar\theta) = \int d\bar\theta d\theta\, \Psi^*_{\text{state}'}(\theta) e^{-\bar\theta\theta} \Psi_{\text{state}}(\bar\theta)$$

$$\boxed{(c_0 + c_1\bar\theta)^* := (c_0^* + c_1^*\theta)}$$

Now. we have enough preparation.

⑥ Express time-evolution in terms of path integral.

$$\Psi(\bar\theta_N @ t_N) = \left(\langle 0| + \bar\theta_N \langle 1|\right) e^{-iHT} |\text{state} @ t_0\rangle$$

$$= \left(\langle 0| + \bar\theta_N \langle 1|\right) e^{-iH(\Delta t)} \int d\bar\theta_{N-1} d\theta_{N-1} \left(|0\rangle + |1\rangle \theta_{N-1}\right) e^{-\bar\theta_{N-1}\theta_{N-1}} \left(\langle 0| + \bar\theta_{N-1} \langle 1|\right)$$

$$e^{-iH\Delta t} \int d\bar\theta_{N-2} d\theta_{N-2} \left(|0\rangle + |1\rangle \theta_{N-2}\right) e^{-\bar\theta_{N-2}\theta_{N-2}} \left(\langle 0| + \bar\theta_{N-2} \langle 1|\right)$$

$$\vdots \qquad \vdots$$

$$e^{-iH\Delta t} \int d\bar\theta_1 d\theta_1 \left(|0\rangle + |1\rangle \theta_1\right) e^{-\bar\theta_1\theta_1} \left(\langle 0| + \bar\theta_1 \langle 1|\right)$$

$$e^{-iH\Delta t} \int d\bar\theta_0 d\theta_0 \left(|0\rangle + |1\rangle \theta_0\right) e^{-\bar\theta_0\theta_0} \Psi_{\text{state}}(\bar\theta_0)$$

Let $\begin{pmatrix} \langle 0|H|0\rangle & \langle 0|H|1\rangle \\ \langle 1|H|0\rangle & \langle 1|H|1\rangle \end{pmatrix} = \begin{pmatrix} H_{00} & H_{01} \\ H_{10} & H_{11} \end{pmatrix} \Rightarrow H = H_{00}\, b b^\dagger + H_{01} b + b^\dagger H_{10} + H_{11} b^\dagger b.$

$$= H_{00} + H_{01} b + b^\dagger H_{10} + (H_{11} - H_{00}) b^\dagger b$$

Evaluate the matrix elements (mod $(\Delta t)^2$ terms.)

$$\left(\langle 0| + \bar\theta_k \langle 1|\right) e^{-iH\Delta t} \left(|0\rangle + |1\rangle \theta_{k-1}\right)$$

$$\cong \left(1 + \bar\theta_k \theta_{k-1}\right) - i(\Delta t)\left(H_{00} + H_{01}\theta_{k-1} + \bar\theta_k H_{10} + H_{11} \bar\theta_k \theta_{k-1}\right)$$

$$\cong \exp\left[\bar\theta_k \theta_{k-1} - i(\Delta t)\ H_{00} + H_{01}\theta_{k-1} + \bar\theta_{k-1} H_{10} + (H_{11} - H_{00}) \bar\theta_k \theta_{k-1}\right]$$

$$= \exp\left[\bar\theta_k \theta_{k-1} - i(\Delta t)\ H\left(b \to \theta_{k-1},\ b^\dagger \to \bar\theta_k\right)\right]$$

So, $\Psi(\bar\theta_N @ t_N) \cong \int d\bar\theta_{N-1} d\theta_{N-1} \cdots d\bar\theta_1 d\theta_1 d\bar\theta_0 d\theta_0$

$$\exp\left[\sum_{k=0}^{N-1} \left(\bar\theta_{k+1} \theta_k\right) - \sum_{k=0}^{N-1} \left(\bar\theta_k \theta_k\right) - i(\Delta t) \sum_{k=0}^{N-1} H\left(\begin{smallmatrix} b \to \theta_k \\ b^\dagger \to \bar\theta_{k+1} \end{smallmatrix}\right)\right] \Psi_{\text{state}}(\bar\theta_0)$$

$$\cong \int d\bar\theta_{N-1} \sim d\theta_0\ \exp\left[i \int dt \left\{(-i\partial_t \bar\theta)\theta - H\left(b \to \theta,\ b^\dagger \to \bar\theta\right)\right\}\right] \Psi_{\text{state}}(\bar\theta_0)$$

$$= \int d\bar\theta_{N-1} \sim d\theta_0\ \exp\left[i \int dt\ \bar\theta(i\partial_t \theta) - H\left(b \to \theta,\ b^\dagger \to \bar\theta\right)\right] \Psi_{\text{state}}(\bar\theta_0)$$

$$\left(\text{cf.}\quad L = \bar\theta(i\partial_t \theta) - H.\right)$$

◎ How to set the bdry condition in path integral after evolution.

just combine the "inner product" and "time evolution" p.4 & p.5.

$$\langle \text{state}' @ t_N | \, e^{-iHT} \, | \text{state} @ t_0 \rangle$$

$$= \int d\bar{\theta}_N d\theta_N \, d\bar{\theta}_{N-1} \sim d\theta_0 \;\; \Psi^*_{\text{state}'}(\theta_N) \, e^{\left[\sum_{k=0}^{N-1} (\bar{\theta}_{k+1}\theta_k - \bar{\theta}_k \theta_k) - i\Delta t \sum_{k=0}^{N-1} H(b \to \theta_k, b^\dagger \to \bar{\theta}_{k+1})\right]} \, \Psi_{\text{state}}(\bar{\theta}_0)$$

◎ More general operator matrix elements in path integral formulation.

Because ...

$$\left(\langle 0| + \bar{\theta}_k \langle 1|\right) e^{-iH\Delta t} \, b \left(|0\rangle + |1\rangle \theta_{k-1}\right) \underset{\text{mod } (\Delta t)^2}{\cong} \exp\left[-i(\Delta t)\left(H_{00} + \bar{\theta}_k H_{10}\right)\right] \theta_{k-1}$$

$$= \exp\left[\bar{\theta}_k \theta_{k-1} - i(\Delta t) H(b \to \theta_{k-1}, b^\dagger \to \bar{\theta}_k)\right] \theta_{k-1}$$

$$\left(\langle 0| + \bar{\theta}_k \langle 1|\right) b^\dagger e^{-iH\Delta t} \left(|0\rangle + |1\rangle \theta_{k-1}\right) \underset{\text{mod } (\Delta t)^2}{\cong} \bar{\theta}_k \exp\left[-i(\Delta t)\left(H_{00} + H_{01}\theta_{k-1}\right)\right]$$

$$= \bar{\theta}_k \exp\left[\bar{\theta}_k \theta_{k-1} - i(\Delta t) H(b \to \theta_{k-1}, b^\dagger \to \bar{\theta}_k)\right]$$

$$\langle \text{state}' @ t_N | \, T\left\{ e^{-i\int dt H} \;\; b^\dagger_{\text{at } t_i} \;\; b_{\text{at } t_j} \right\} | \text{state} @ t_0 \rangle \qquad \text{(Schrödinger picture)}$$

$$= \langle \text{state}' | \, T\{ b^\dagger(t_i) \, b(t_j) \} \, | \text{state} \rangle \qquad \text{(Heisenberg picture)}$$

$$= \int d\bar{\theta}_N \sim d\theta_0 \;\; \Psi^*_{\text{state}'}(\theta_N) \, e^{i\int dt \left\{ \bar{\theta}(i\partial_t \theta) - H(b \to \theta, b^\dagger \to \bar{\theta}) \right\}} \, \bar{\theta}_i \theta_j \;\; \Psi_{\text{state}}(\bar{\theta}_0)$$

§ 1.3 Path integral formulation for quantum field theories

$\boxed{\text{Example 1:}}$ a free scalar boson on $(d+1)$-dimensional space-time

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 \iff \mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\partial_i\phi)(\partial_i\phi) + \frac{1}{2}m^2\phi^2$$

$$\left[\phi(\vec{x},t),\ \pi(\vec{y},t)\right] = i\delta^d(\vec{x}-\vec{y})$$

This is just a quantum mechanichs of infinitely many harmonic oscillators.

Individual oscillators are labeled by $\vec{k}$ $\left(\omega = \sqrt{\vec{k}^2+m^2}\right)$.

$$\boxed{\begin{aligned}&\langle\Omega|T\{\phi(x_1^\mu)\,\phi(x_2^\mu)\cdots\ \pi(y_1^\mu)\,\pi(y_2^\mu)\cdots\}|\Omega\rangle\\ &= \frac{\int_{+i\varepsilon}\mathcal{D}\phi\,\mathcal{D}\pi\ e^{i\int dt\int d^dx\,(\pi\dot{\phi}-\mathcal{H}=\mathcal{L})}\left(\phi(x_1)\phi(x_2)\cdots\pi(y_1)\pi(y_2)\cdots\right)}{\int_{+i\varepsilon}\mathcal{D}\phi\,\mathcal{D}\pi\ e^{i\int dt\int d^dx\,\mathcal{L}}}\end{aligned}}$$

Example 1′: add interaction terms to $\mathcal{H}$ $\left(\text{eg. } += \frac{\lambda}{4!}\phi^4\right)$

$\Rightarrow$ just replace $(\pi\dot{\phi}-\mathcal{H}_0)$ by $(\pi\dot{\phi}-\mathcal{H})$

$\boxed{\text{Example 2:}}$ Dirac theory $\mathcal{L} = \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi$

$$\boxed{\begin{aligned}&\langle\Omega|T\{\Psi(x_1)\Psi(x_2)\cdots\ \bar{\Psi}(y_1)\bar{\Psi}(y_2)\cdots\}|\Omega\rangle\\ &= \frac{\int_{+i\varepsilon}\mathcal{D}\bar{\Psi}\mathcal{D}\Psi\ e^{i\int dt\,d^dx\,\mathcal{L}}\left(\Psi(x_1)\Psi(x_2)\cdots\bar{\Psi}(y_1)\bar{\Psi}(y_2)\cdots\right)}{\int\mathcal{D}\bar{\Psi}\mathcal{D}\Psi\ e^{i\int dt\,d^dx\,\mathcal{L}}}\end{aligned}}$$

where we integrate over infinitely many Grassmann coordinates.

$\hookrightarrow$ (labeled by $\vec{k}\in\mathbb{R}^3$, Ψ vs $\bar{\Psi}$ and time slices.)