§ 2 What is QFT (for)?

probably too early to ask the question.

Quantum mechanics : a vector space $\mathcal{H}$ over $\mathbb{C}$
with a positive definite Hermitian inner product on $\mathcal{H}$.
set of linear operators acting on $\mathcal{H}$.

Quantum Mechanics [set of tools (procedures). thinking framework] Quantum Field Theory so?

systems	applicable to	models.
(free particle, harmonic oscill., diatomic molecule, ⋮ finite DOF's.) $\Leftarrow$	(many concrete set-ups.) $\Rightarrow$	[individual QFT's] ((see below))

A couple of characterizations of QFT

★ a perspective in a quantum many-body system

A quantum mechanical system of N fermions with.

$$H = \sum_{i=1}^{N}\left(-\frac{1}{2m}\partial_{x_i}^2 + \varphi(x_i)\right) + \sum_{i\neq j}^{N}\frac{C}{|x_i - x_j|}$$ is "equivalent" to

~~a QFT model with~~

$\psi(x) := \Sigma_n \psi_n(x)\, a_n$. ann. operator $\qquad \psi^\dagger(x) := \Sigma_n \psi_n^*(x)\, a_n^\dagger$ creat'n op.

$\{a_n, a_m^\dagger\} = \delta_{n,m} \qquad \{\psi(\vec{x}), \psi^\dagger(\vec{y})\} = \delta(\vec{x}-\vec{y})$

$$H = \int dx\, \psi^\dagger(x)\left(-\frac{\partial_x^2}{2m} + \varphi(x)\right)\psi(x) + \int dx \int dy\, \psi^\dagger(y)\psi(y)\frac{C}{|x-y|}\psi^\dagger(x)\psi(x).$$

dictionary :

The many body wavefunct'n $\Psi(x_1, x_2, \cdots, x_N) = \frac{1}{\sqrt{N!}} \begin{vmatrix} \psi_{n_1}(x_1) & \cdots & \psi_{n_1}(x_N) \\ \vdots & \ddots & \vdots \\ \psi_{n_N}(x_1) & \cdots & \psi_{n_N}(x_N) \end{vmatrix}$ of a $|\text{state}\rangle$
$\overset{\|}{=}\ a^\dagger_{n_1} a^\dagger_{n_2} \cdots a^\dagger_{n_N}|0\rangle$

is an operator matrix element $\langle 0|\psi(x_N)\cdots\psi(x_2)\psi(x_1)|\text{state}\rangle$.

$\Rightarrow$ can think of a broader class of observables. — (*1)

$\Psi(x_1, \cdots, x_N)$: regarded also as a result of comparing against a set of states $\left(\langle 0|\psi(x_N)\cdots\psi(x_1)\right)_{x_1 \sim x_N \in (\mathbb{R}^d)^N}$.

measurement : comparison against a well-established "things"

$\Longrightarrow$ can be against any basis of $\mathcal{H}$ (the vector space of all the states)

$\langle e_i|\text{state}\rangle$ for $\text{Span}\{|e_i\rangle\} = \mathcal{H}$. (*2)

The probability interpretation of $\Psi(x_1, \cdots, x_N) = \langle 0|\psi(x_N)\cdots\psi(x_1)|\text{state}\rangle$

is available only when the particle number is conserved.

The QFT description ((*1), (*2)) does more than the QM description using $\Psi(x_1, \cdots, x_N)$.

The QFT approach to a many-body system often focuses on each one of particles interacting with all others instead of studying the entire system.

☆ Quantum theory of dynamical systems where a finite number of degrees of freedom are attributed to each point in space.
(a subclass of QM systems with infinitely many DOF's)

models

- ψ : fermion. $H = \int dx\, \psi^\dagger \left(\frac{-\nabla^2}{2m} + V(x)\right)\psi + \int dx \int dy\, (\psi^\dagger\psi)(y) \frac{c}{|x-y|} (\psi^\dagger\psi)(x).$

$$L = \int dx\, \psi^\dagger (i\partial_t \psi) - H$$

- ϕ : complex scalar. $\mathcal{L} = \phi^\dagger \left(i\partial_t + \frac{\partial^2}{2m} - V(\vec{x})\right)\phi - f(|\phi|^2)$ → Translational sym. on space is not mandatory in QFT.

  ~~e.g. cold atom system in a trap potential~~

- ~~Hubbard model~~ $L = \int dx \sum_{\sigma = \uparrow,\downarrow} \psi_\sigma^\dagger (i\partial_t \psi_\sigma) - H$

$$H = \int dx \left\{ \sum_{\sigma=\uparrow\downarrow} \left(\psi_\sigma^\dagger \frac{-\nabla^2}{2m}\psi_\sigma\right) + U(\psi_\uparrow^\dagger\psi_\uparrow)(\psi_\downarrow^\dagger\psi_\downarrow)\right\}$$

$\psi_{\sigma=\uparrow,\downarrow}$: 2-component fermion

- Klein-Gordon model. $L = \int dx \left\{ \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 - V(\phi)\right\}$

  There is nothing wrong in the quantum theory of the Klein-Gordon model.

  Just do the canonical quantization.

- electron in a periodic potential (crystal)

$$\psi_\sigma(x) = \Sigma_n \psi_n(x)\, a_n \rightarrow \int_{\text{Brillouin zone (B.Z.)}} \frac{d^d k}{(2\pi)^d} \sum_{b \in \text{bands}} \psi_b(x;k)\, a_{b,k,\sigma}$$

σ: spin $\uparrow,\downarrow$

$\psi_b(x;k)$: Bloch wavefunction of a state in the band "b" with the momentum $k \in$ B.Z.

multiple bands:
may define multiple field operators

$$\psi_{b,\sigma}(x) := \int \frac{d^d k}{(2\pi)^d} \psi_b(x;k)\, a_{b,k,\sigma}.$$

- electron-phonon system
- spin system.

· the Standard model of particle physics.

(1+3+8) vector fields. 45 2-component fermion fields

2 complex scalar fields.

· Quantum Electrodynamics

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \Sigma_j \bar{\Psi}_j \left\{ i \gamma^\mu (\partial_\mu + ieQ_j A_\mu) - m_j \right\} \Psi_j$$

Ψ_j : 4-component fermion field

· nucleon – meson system

⋮

Think of those models as dynamical systems with some DOF's labeled by points in space. Then follow the ordinary procedure of quantization. (use Dirac brackets if necessary : hw VI-1)

For those models (systems), we will wish to work out :

- what the ground state is like
- what the excitation spectrum is like
- how the system/states evolve in time
- how various operator expectation values are correlated

would be nice to have a tool box / thinking framework. that can be applied to a broad class of models.

☆ a tool. (procedure) to compute the "rates" of processes where the number of particles / particle spieces is not preserved

e.g. $e^+ + e^- \to 2\gamma$. $e^+ + e^- \to q + \bar{q}$ (quark–anti quark)

$g + g \to h + t + \bar{t}$

(proton) + (proton) → (many hadrons)

photo-emission / absorption. ---

☆ a general theory constraining quantum systems on (space ×(time)) with basic principles.

· The Hilbert space ((physicists dialect) := the vector space of all the states in a quantum system in consideration.) does not necessarily look like a Fock space.

(a vector space spanned by {|vacuum⟩, 1-particle excitat'n on |vac⟩, 2-particle excitat'n on |vac⟩, ...})

& only in a system that can be regarded as that of weakly interacting particles.

· A Lagrangian is not always available.

· basic principles :

locality, causality, unitarity ...

How far can we go with them ?

Supplementary notes on spinor fields on $\mathbb{R}^{d,1}=\mathbb{R}^{3,1}$ Minkowski space-time.

§A

For any representation ρ of the group $SO(3,1)$, we can think of a field

$\phi(x^\mu)$ whose Lorentz transformation is given by

$$\phi(x^\mu) \longrightarrow \rho(g)\cdot\phi((g^{-1}\cdot x)^\mu) \qquad g\in SO(3,1)$$

scalar fields: $\phi(x^\mu) \longrightarrow \phi(y^\mu)$ $\quad$ ρ: trivial repr.

vector fields: $\phi_\lambda(x^\mu) \longrightarrow g_\lambda{}^\kappa\,\phi_\kappa(y^\mu)$. $\quad$ ρ: defining repr.

$$y^\mu=(g^{-1})^\mu{}_\nu x^\nu$$

There are other representations of $Spin(3,1)$

$\longrightarrow$ there are other kinds of fields.

Let $\eta^{\mu\nu}=\mathrm{diag}(+,-,-,-)$ and $\gamma^{\mu=0}=\begin{pmatrix}0 & \mathbb{1}_{2\times2}\\ \mathbb{1}_{2\times2} & 0\end{pmatrix}$, $\gamma^{\mu=i}=\begin{pmatrix}0 & \tau^i\\ -\tau^i & 0\end{pmatrix}$. $\Rightarrow \{\gamma^\mu,\gamma^\nu\}=2\eta^{\mu\nu}$

Then for $\Lambda^\rho{}_\sigma=\begin{bmatrix}0 & \beta^1 & \beta^2 & \beta^3\\ \beta^1 & 0 & \theta^3 & -\theta^2\\ \beta^2 & -\theta^3 & 0 & \theta^1\\ \beta^3 & \theta^2 & -\theta^1 & 0\end{bmatrix}$ $\qquad (g)^\kappa{}_\lambda=(\exp[\Lambda])^\kappa{}_\lambda$

$$\left(\Rightarrow \frac{1}{4}\Lambda_{\rho\sigma}[\gamma^\rho,\gamma^\sigma]=\begin{pmatrix}-\vec\beta\cdot\vec\tau+i\vec\theta\cdot\vec\tau & 0\\ 0 & \vec\beta\cdot\vec\tau+i\vec\theta\cdot\vec\tau\end{pmatrix}\right)$$

$$\rho_{spin}(g):=\exp\left(\frac{1}{8}\Lambda_{\rho\sigma}[\gamma^\rho,\gamma^\sigma]\right)=\begin{pmatrix}\exp[-\frac{\vec\beta}{2}\cdot\vec\tau+i\frac{\vec\theta}{2}\cdot\vec\tau] & 0\\ 0 & \exp[+\frac{\vec\beta}{2}\cdot\vec\tau+i\frac{\vec\theta}{2}\cdot\vec\tau]\end{pmatrix}$$

$$=:\begin{pmatrix}\rho_L(g) & 0\\ 0 & \rho_R(g)\end{pmatrix}$$

ρ_{spin}, ρ_L, ρ_R are all representations of $Spin(3,1)$.

4-component (Dirac) spinor fields $\quad \rho=\rho_{spin}$	$\Psi(x)\to\rho_{spin}(g)\Psi(y)$
2-component: left handed spinor fields : $\rho=\rho_L$	$\psi(x)\to\rho_L(g)\psi(y)$
2-component: right handed spinor fields : $\rho=\rho_R$	$\chi(x)\to\rho_R(g)\chi(y)$

$(y^\mu=(g^{-1})^\mu{}_\nu x^\nu)$

§B The Dirac Lagrangian

*D1

$$\mathcal{L}_{\text{Dirac}} = \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi$$ is known to be invariant under the Lorentz transformation.

$$\left(\bar{\Psi} := \Psi^\dagger\gamma^0\right)$$

γ^μ's : 4×4 matrices

$\Psi(x)$: 4-component spinor fields.

It is also possible to use a notation

$$\underset{\text{4-component}}{\Psi} =: \begin{pmatrix}\psi\\ \chi\end{pmatrix} \quad \text{both 2-component.} \qquad \bar{\Psi} = (\psi^\dagger, \chi^\dagger)\begin{pmatrix} & 1_{2\times2}\\ 1_{2\times2} & \end{pmatrix} = (\chi^\dagger, \psi^\dagger)$$

*D2

so $$\boxed{\mathcal{L}_{\text{Dirac}} = \psi^\dagger i\bar{\sigma}^\mu(\partial_\mu\psi) + \chi^\dagger i\sigma^\mu(\partial_\mu\chi) - m\,\chi^\dagger\psi - m\,\psi^\dagger\chi .}$$

All the 4 terms are known to be individually Lorentz invariant.

$$\bar{\sigma}^\mu = (1_{2\times2}, -\vec{\tau}) \qquad \sigma^\mu = (1_{2\times2}, \vec{\tau})$$

$\mu = 0,\ 1\sim3$ $\qquad$ $\mu = 0,\ 1\sim3$

lower-left block of γ^μ $\qquad$ upper-right block of γ^μ.

There is another way to write $\mathcal{L}_{\text{Dirac}}$.

Mathematically

$$[\rho_R(g)]^{c.c.} = \begin{pmatrix} & -1\\ 1 & \end{pmatrix}\rho_L(g)\begin{pmatrix} & 1\\ -1 & \end{pmatrix}$$

(complex conjugation on each matrix elements)

$$[\rho_L(g)]^{c.c.} = \begin{pmatrix} & 1\\ -1 & \end{pmatrix}\rho_R(g)\begin{pmatrix} & -1\\ 1 & \end{pmatrix}$$

as 2×2 matrices.

$\Rightarrow$ for a right handed spinor field χ

$$\begin{pmatrix} & 1\\ -1 & \end{pmatrix}\chi^* =: \psi_\chi \quad \text{is a left-handed spinor field.}$$

for a left-handed spinor field ψ.

$$\begin{pmatrix} & -1\\ 1 & \end{pmatrix}\psi^* =: \chi_\psi \quad \text{is a right-handed spinor field.}$$

So, it is also possible to use a notation

$$\Psi = \begin{pmatrix} \psi \\ \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} (\psi_\chi)^* \end{pmatrix} \qquad \bar{\Psi} = \left(\psi_\chi^T \cdot \begin{pmatrix} & 1 \\ -1 & \end{pmatrix},\ \psi^\dagger \right)$$

$$\boxed{\mathcal{L}_{Dirac}} = \psi^\dagger i \bar{\sigma}^\mu (\partial_\mu \psi) + \psi_\chi^T i \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \sigma^\mu \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} (\partial_\mu \psi_\chi^*)$$
$$- m\, \psi_\chi^T \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \psi - m\, \psi^\dagger \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \psi_\chi^*$$

$$= \psi^\dagger i \bar{\sigma}^\mu (\partial_\mu \psi) + \psi_\chi^T i (\bar{\sigma}^\mu)^T (\partial_\mu \psi_\chi^*)$$
$$- m\, \psi_\chi^T \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \psi - m\, \psi^\dagger \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \psi_\chi^*$$

$$\boxed{\begin{aligned} &= \psi^\dagger i \bar{\sigma}^\mu (\partial_\mu \psi) + \psi_\chi^\dagger i \bar{\sigma}^\mu (\partial_\mu \psi_\chi) \\ &\quad - m\, \psi_\chi^T \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \psi - m\, \psi^\dagger \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \psi_\chi^* \end{aligned}} \quad (*D3)$$

All the 4 terms are individually Lorentz invariant.

This Dirac Lagrangian (*D1) = (*D2) = (*D3) describes physics of e^- and e^+ (or μ^- and μ^+) when $m = 511$ keV ($m = 105$ MeV)

§C The neutrino kinetic term

Just use one 2-component spinor field ψ

$$\boxed{\mathcal{L}_\nu = \psi^\dagger i \bar{\sigma}^\mu (\partial_\mu \psi) - \frac{m}{2} \psi^T \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \psi - \frac{m}{2} \psi^\dagger \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \psi^*} \quad (*D1)$$

$\begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \psi^*$ is a right-handed spinor field.

(*D1) is obtained from (*D3) by setting $\psi_\chi = \psi$ and rescaling $\psi \to \frac{1}{\sqrt{2}} \psi$. (field redefinition)

(e^+ and e^- are different so $\psi_\chi \neq \psi$.
there is just ν $\rightarrow$ set $\psi_\chi = \psi$.)

There is another way to write $\mathcal{L}_\nu$.

$$\Psi_\nu := \begin{pmatrix} \psi \\ \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\psi^* \end{pmatrix} \qquad \bar{\Psi} = \left(\psi^T \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \psi^\dagger\right)$$

$$\boxed{\mathcal{L}_\nu = \frac{1}{2}\bar{\Psi}_\nu i\gamma^\mu(\partial_\mu \Psi_\nu) - \frac{m}{2}\bar{\Psi}_\nu\Psi_\nu} \quad (*\nu 2)$$

(*ν2) is obtained from (*D1) by

~~imposing a condition~~ $\boxed{\begin{pmatrix} & \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \\ \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} & \end{pmatrix}\Psi^* = \Psi}$ (Majorana condition)

$\left(\Leftrightarrow \psi_\chi = \psi \Leftrightarrow \chi = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\psi^*\right)$

and rescaling the field

$(\Psi_{\text{w/ Maj. cond}}) \to \frac{1}{\sqrt{2}}\Psi_\nu$

$\boxed{\S D}$ The non-relativistic limit of $\mathcal{L}_{\text{Dirac}}$

→ ~~described by a 2-component spinor field~~ ψ_e.

↳ not the same as ψ or ψ_χ or χ.

in $\Psi = \begin{pmatrix} \psi \\ \chi \end{pmatrix}$

⇝ (see week 07 lecture note
~~hw VII-1. and XIV-1.~~)

$\boxed{\S E}$ In condensed matter applications,

- electrons might be dealt with by multiple fields (see p.3 ~~this week~~)
- low-dim modes (e.g. surface modes) may or may not carry full $SO(3) \subset SO(3,1)$ spin DOF.

~~§2~~ §3. S-matrix etc. and how to compute them.

~~§2.1~~ §3.1 S-matrix, decay rate and cross section

Think of a case where interactions are turned on after $t=T_-$ and switched off by $t=T_+$.

H : the full Hamiltonian

H_0 : the bilinear part of H_0 $\qquad H = H_0 + V$

example (QED) $\qquad H_0 = -\bar{\Psi}(i\gamma^i\partial_i - m)\Psi + (\text{photon})$

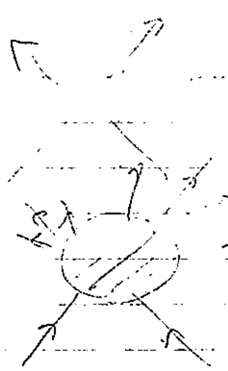
$$V = Q_\Psi \bar{\Psi}(\gamma^i e A_i + \gamma^0 e A_0)\Psi$$

The Hilbert space of the free H_0 theory is generated by Fock states

$$\{\,|0\rangle,\ a^\dagger_{n,\vec{p}}|0\rangle,\ a^\dagger_{n_1,\vec{p}_1} a^\dagger_{n_2,\vec{p}_2}|0\rangle,\ \cdots\ \} \quad \text{at } t=t_*.$$

For the interacting theory $(H_0 \rightsquigarrow H \rightsquigarrow H_0)$, we can use

H_0



the in-states

$$e^{\left(-i\int_{T_-}^{t_*} dt' H\right)} e^{-iH_0(T_- - t_*)} \text{ on } \{\,|0\rangle,\ a^\dagger_{n,\vec{p}}|0\rangle,\ a^\dagger_{n_1,\vec{p}_1} a^\dagger_{n_2,\vec{p}_2}|0\rangle, \cdots\}$$

superpos

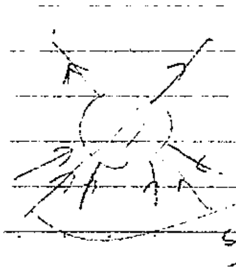
as a basis of the Hilbert space.

The "in-states" are denoted by

$$|\Omega\rangle,\quad \underset{\propto |n,\vec{p}\rangle^{in}}{a^\dagger_{n,\vec{p}}|\Omega\rangle} \quad \underset{\propto |n_1,\vec{p}_1; n_2,\vec{p}_2\rangle^{in}}{a^\dagger_{n_1\vec{p}_1} a^\dagger_{n_2\vec{p}_2}|\Omega\rangle}$$

We can also take another basis :

the out-states



$$e^{\left(i\int_{T_+}^{t_*} dt' H\right)} e^{-iH_0(T_+ - t_*)} \text{ on } \{\,|0\rangle,\ a^\dagger_{n,\vec{p}}|0\rangle,\ a^\dagger_{n_1,\vec{p}_1} a^\dagger_{n_2,\vec{p}_2}|0\rangle, \cdots\},$$

superpos.

The "in-states" and "out-states" are normalized so that

$$\boxed{\begin{aligned} {}^{in}\langle n_1, \vec{p}_1 | n_2, \vec{p}_2 \rangle^{in} &= \delta_{n_1,n_2} (2E_{\vec{p}_1}) \delta^3(\vec{p}_1 - \vec{p}_2)(2\pi)^3 \\ {}^{out}\langle n_1, \vec{p}_1 | n_2, \vec{p}_2 \rangle^{out} &= \delta_{n_1,n_2} (2E_{\vec{p}_1}) \delta^3(\vec{p}_1 - \vec{p}_2)(2\pi)^3 \end{aligned}}$$

in relativistic situations. (if non-rela. ${}^{in}\langle \vec{p} | \vec{q} \rangle^{in} = \delta^3(\vec{p} - \vec{q})(2\pi)^3$ is more conventional.)

The S-matrix is defined by

$$\boxed{S_{\beta\alpha} := {}^{out}\langle \beta | \alpha \rangle^{in},}$$

α, β: Fock states

which measures the difference between the in-state basis and the out-state basis.

The truely scattering part of the S-matrix is in. $S_{\beta\alpha} = \mathbb{1}_{\beta\alpha} + (2\pi)^4 \delta^4(p_{out} - p_{in})\, i\mathcal{M}_{\beta\alpha}$.

For a single particle : decay rate

$$\boxed{d\Gamma = \frac{1}{(2E_{in})} \prod_{i=1}^{N_f} \left[\frac{d^3\vec{p}_i}{(2\pi)^3} \frac{1}{2E_{\vec{p}_i}} \right] (2\pi)^4 \delta^4(p_f - p_{in}) |\mathcal{M}|^2}$$

Lorentz invariant

A boosted particle has a longer lifetime.

For a pair of colliding particles : cross section

$$\boxed{d\sigma = \frac{1}{(2E_1)(2E_2)|\vec{v}_1 - \vec{v}_2|} \prod_{i=1}^{N_f} \left[\frac{d^3\vec{p}_i}{(2\pi)^3} \frac{1}{2E_{\vec{p}_i}} \right] (2\pi)^4 \delta^4(p_f - p_{in}) |\mathcal{M}|^2}$$

$|\vec{v}_1 - \vec{v}_2| := \left| \frac{p_1^z}{E_1} - \frac{p_2^z}{E_2} \right|$ when the two particles are moving ~~in the dire~~ along the z-axis.

boost invariant.

Q: Verify that $[P]=+1$ and $[\sigma]=-2$ for arbitrary N_f by dimension counting.

Q: Verify that $E_1 E_2 |\vec{v}_1-\vec{v}_2|$ is boost invariant. (along the collision axis).

Notes

$\frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}$ is boost invariant.

This is because $\begin{pmatrix} p_z' \\ E' \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix}\begin{pmatrix} p_z \\ E \end{pmatrix}$, and $\frac{dp_z'}{E'} = \frac{dp_z\,\gamma\,(1+\frac{p_z}{E}\beta)}{E'} = \frac{dp_z}{E}$.

$(2E_{\vec{p}})\,\delta^3(\vec{p}-\vec{q})$ is also invariant.

A side remark

π^+ : $m \simeq 140$ MeV, $\tau \simeq 2.6\times10^{-8}$ s. $\pi^+ \to \mu^+ + \nu_\mu$ (2-body)

μ^+ : $m \simeq 105$ MeV, $\tau \simeq 2.2\times10^{-6}$ s $\mu^+ \to e^+ + \nu_e + \bar{\nu}_\mu$ (3-body)

§3.2. LSZ reduction formula

relates time-ordered correlations with the S-matrix.

§3.2.1 Källen–Lehmann spectral representation

When a QFT (model) is built from ($\mathcal{H} \simeq$ Fock space) + interactions.

then $\exists$ a clear relation between fields (operators) and 1-particle states.

$$\Psi(x^\mu) := \sum_n \psi_n(x^\mu)\, a_n \qquad \langle 0 | \Psi(x) | n \rangle = \psi_n(x)$$

When a QFT model is given first, on the other hand,

we do not know a priori which operators are associated with

a one particle state (even when a Fock space structure is in the Hilbert space).

Example: QED: easy. $\Psi \leftrightarrow e^-$
$\Psi^\dagger \leftrightarrow e^+$
$A_\mu \leftrightarrow \gamma$

QCD: given by gluon field A_μ^a, quark fields $\Psi, \bar{\Psi}$.
but $\pi^\pm, \pi^0, \rho \cdots$ come out as 1-particle states at low-energy.

Quantum Hall system: similar issues.

A necessary condition for one particle states $\subset \mathcal{H}$ (Hilbert space)

a family of states $|X(\vec{p})\rangle$, labeled by $\vec{p} \in \mathbb{R}^3$.

with $H\,|X(\vec{p})\rangle = \sqrt{(m_X^2 + \vec{p}^{\,2})}\;|X(\vec{p})\rangle$ (assuming $SO(3,1)$ symmetry in this discussion)

for some m_X (indep. of $\vec{p}$) $\in \mathbb{R}_{\geq 0}$

An operator in a QFT model is

said to be "associated with" one particle states $|X(\vec{p})\rangle$

when ... (next page)

Consider

$$\int d^4y \int d^4x \; e^{ip\cdot y} e^{-iq\cdot x} \langle \Omega | T\{A(y)\; B(x)\} | \Omega \rangle . \quad \text{——} (\ast)$$

$|\Omega\rangle$: the vacuum state in a interacting QFT model.

$A(x), B(x)$: operators in that QFT model.

Assume translational symmetry on $\mathbb{R}^{1,3}$

$\Leftrightarrow$ Noether change P^μ well-defined on $\mathcal{H}$.
and $P^\mu|\Omega\rangle = 0$.

$$\langle \Omega | T\{A(y)\, B(x)\} | \Omega \rangle$$
$$= \langle \Omega | e^{-iP\cdot a} T\{A(y)\, B(x)\} e^{iP\cdot a} | \Omega \rangle$$
$$= \langle \Omega | T\{ e^{-iP\cdot a} A(y) e^{iP\cdot a} \; e^{-iP\cdot a} B(x) e^{iP\cdot a} \} | \Omega \rangle$$
$$= \langle \Omega | T\{ A(y-a)\; B(x-a)\} | \Omega \rangle \quad \text{for any } a^\mu \in \mathbb{R}^{1,3}.$$

So $\langle \Omega | T\{A(y)\,B(x)\}|\Omega\rangle$ depends only on $(x-y)^\mu$.

$$\Rightarrow (\ast) = (2\pi)^4 \delta^4(p-q) \int d^4z \; e^{ip\cdot z} \langle \Omega | T\{ A(z)\; B(0)\} | \Omega \rangle .$$

How does $\underbrace{\int d^4z \; e^{ip\cdot z} \langle \Omega | T\{A(z)\, B(0)\}|\Omega\rangle}_{(\ast\ast)}$ depend on p^μ ?

A partial contribution to $(\ast\ast)$
from integration over $z^0 \in$ (positive, $+\infty$) region
with $|X(\vec{p})\rangle\langle X(\vec{p})|$ insertion.

$$\underset{X}{\Delta}(\ast\ast) = \int_{0+}^{+\infty} dz^0 \int d^3z \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2E_{\vec{k}}} e^{ip\cdot z} \langle \Omega | A(z) | X(\vec{k})\rangle\langle X(\vec{k}) | B(0) | \Omega \rangle$$

$$= \int_{0+}^{+\infty} dz^0 \int d^3z \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{(2E_{\vec{k}}(X))} e^{ip\cdot z} \langle \Omega | \overset{\text{Noether change } P^\mu \text{ operator.}}{e^{iP\cdot z}} A(0)\, e^{-iP\cdot z} | X(\vec{k})\rangle\langle X(\vec{k}) | B(0) | \Omega \rangle$$

$$= \int_{0+}^{+\infty} dz^0 \int d^3z \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{(2E_{\vec{k}})} e^{ip\cdot z} e^{-i\underline{k}\cdot z} \langle \Omega | A(0) | X(\vec{k})\rangle\langle X(\vec{k}) | B(0) | \Omega \rangle$$

$k^0 = \sqrt{m_X^2 + \vec{k}^2}$ ($\underline{k}$: on-shell condition imposed)

PLUS

$d^3 z\ d^3 k$ integrat'n

$$\Delta_X(※※) = \cdots = \int_{0+}^{+\infty} dz^0 \frac{e^{i(p^0 - E_{\vec p})z^0}}{(2E_{\vec p})} \langle\Omega| A(0) |X(\vec p)\rangle\langle X(\vec p)| B(0) |\Omega\rangle$$

The $(i\varepsilon)$ prescription ← (week 02 lecture)

$\boxed{\text{time integration chosen in the direction } (-\infty, +\infty)\times(1-i\varepsilon) \text{ to obtain } \langle\Omega| T\{\cdots\} |\Omega\rangle.}$

for real valued $p^0 < E_{\vec p}$ the z^0 integration is convergent.

$$= \frac{i}{(2E_{\vec p})(p^0 - E_{\vec p})} \langle\Omega| A(0) |X(\vec p)\rangle\langle X(\vec p)| B(0) |\Omega\rangle.$$

Similarly the integration over $z^0 \in (-\infty, \text{negative})$ contributes by

$$\Delta_X(※※) = \int_{-\infty}^{0-} dz^0 \int d^3 z \int \frac{d^3 k}{(2\pi)^3} \frac{1}{(2E_{\vec k})} e^{ip\cdot z} e^{ik\cdot z} \langle\Omega| B(0) |X(\vec k)\rangle\langle X(\vec k)| A(0) |\Omega\rangle$$

$$= \int_{-\infty}^{0-} dz^0 \frac{e^{i(p^0 + E_{\vec p})z^0}}{(2E_{\vec p})} \langle\Omega| B(0) |X(-\vec p)\rangle\langle X(-\vec p)| A(0) |\Omega\rangle$$

The (z^0) integration toward $-\infty(1-i\varepsilon)$ is convergent for real valued $(-E_{\vec p}) < p^0$.

$$= \frac{i}{(-2E_{\vec p})(p^0 + E_{\vec p})} \langle\Omega| B(0) |X(-\vec p)\rangle\langle X(-\vec p)| A(0) |\Omega\rangle.$$

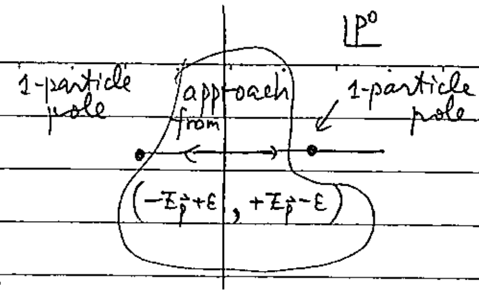
Källen – Lehmann spectral representation:

The time-ordered correlation function of two operators in the Fourier space (※※) is expressed as a sum over excitated states "associated" w/ the operators.

$$\left(\langle \text{excitated state} \overset{\mathbb{1}(\text{def})}{|} \underset{\text{or } B}{A(0)} |\Omega\rangle \neq 0 \right)$$

$$\boxed{\begin{array}{c} \text{A pole in the } p^0\text{-plane} \Leftrightarrow \text{an associated one particle state} \\ \text{branch cuts} \Leftrightarrow \text{multi particle states} \end{array}}$$

A technical note



We have arrived at the picture
of finding poles on the real p^0 axis
when approached from the open interval $(-E_{\vec p}+\varepsilon, +E_{\vec p}-\varepsilon)$.
[in the derivation in the previous 2 pages]

~~That is roughly right, but we can be alittle more precise:~~

the inverse Fourier transform of Δ_X(※※) should reproduce
the $\begin{Bmatrix} e^{-iE_{\vec p}(z^0)} & (z^0>0) \\ e^{+iE_{\vec p}(z^0)} & (z^0<0) \end{Bmatrix}$ contributions.

~~For that property to be satisfied,~~

pole of (※※) at $\begin{Bmatrix} p^0 = (+E_{\vec p} - i\varepsilon) \\ p^0 = (-E_{\vec p} + i\varepsilon) \end{Bmatrix}$ ~~inv. F.T.~~ contour $(p^0) \in \mathbb{R}$.

or

pole of (※※) at $\begin{Bmatrix} p^0 = +E_{\vec p} \\ p^0 = -E_{\vec p} \end{Bmatrix}$ inv. F.T. contour $(p^0) \in \mathbb{R} \times (1+i\varepsilon)$

No.

Date

When the operators $A(x)$, $B(y)$ are associated with a one-particle state $|X(\vec{p})\rangle$

☆ the pole of $\langle X X\rangle$ comes with a residue $Z_X = \langle\Omega|A(0)|X(\vec{p})\rangle\langle X(\vec{p})|B(0)|\Omega\rangle$

The states $B(0)|\Omega\rangle/\sqrt{Z_X}$ contain one particle states $|X(\vec{p})\rangle$

$\langle\Omega|A(0)/\sqrt{Z_X}$ $\langle X(\vec{p})|$

with the appropriate normalization.

☆ Examples:

- free scalar particle: $A(x) = B(x) = \int\frac{d^3k}{(2\pi)^3}\frac{1}{(2E_{\vec{k}})}\left(a_{\vec{k}}e^{-ik\cdot x} + a_{\vec{k}}^\dagger e^{ik\cdot x}\right)$

$\Rightarrow Z = 1.$

- weakly interacting fermions

$$L = \int d^dx\, \psi^\dagger\left(i\partial_t + \frac{\nabla^2}{2m}\right)\psi + \int d^dx\, d^dy\, (\psi^\dagger\psi)(x)\,\frac{c}{|x-y|}(\psi^\dagger\psi)(y)$$

$\psi(x) = \sum_n \psi_n(x)\, a_n$ if $c = 0$.

When $c \neq 0$ but small.

F.T. $\left(\langle\Omega|T\{\psi(x)\psi^\dagger(y)\}|\Omega\rangle\right)$ will still have contrib'n from fermionic one particle states $\left(\text{as in the case of } c=0\right)$

but we should not expect $Z = 1$.

- QCD.

$$\langle\rho \text{ meson}|\sum_{f=u,d}\bar{\Psi}_f\gamma^\mu\Psi_f|\Omega\rangle \neq 0.$$

$$\langle\pi^a \text{ meson}|\sum_{f,f'=u,d}\left(\bar{\Psi}_f\gamma_5(\tau^a)_{ff'}\Psi_{f'}\right)|\Omega\rangle \neq 0.$$

⇑

quark bilinear operators of QCD.

That is how one particle states in interacting models are characterized.

(intuition backed up by experiments) translated to (poles in op-op correlat'n fcns).

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