## §3.2.2 The LSZ Reduction Formula

(LSZ: Lehmann – Symanzik – Zimmermann)

Consider

$$\int d^4x_1 \cdots d^4x_n \; e^{-ip_1\cdot x_1} \cdots e^{-ip_k\cdot x_k} \; e^{+iq_{k+1}\cdot x_{k+1}} \cdots e^{+iq_N\cdot x_N} \qquad (※)$$

$$\langle\Omega| T\{ \mathcal{O}_{n_1}(x_1^\mu)\, \mathcal{O}_{n_2}(x_2^\mu) \cdots \mathcal{O}_{n_N}(x_N^\mu)\} |\Omega\rangle ,$$

where $p_1^{\mu=0} \sim p_k^{\mu=0} > 0$. $q_{k+1}^{\mu=0} \sim q_N^{\mu=0} > 0$.

Choose operators $\mathcal{O}_{n_i}(x)$ that are associated with one particle states $\langle n_i; \vec{p}|$ of particle spieces $n_i$

$$\langle n_i; \vec{p}|\, \mathcal{O}_{n_i}(x) |\Omega\rangle \neq 0 ; \text{ residue } Z_{n_i}.$$

Then.

$$(※) \sim \frac{iZ_{n_1}}{(p_1^2 - m_1^2 + i\epsilon)} \cdots\cdots \frac{iZ_{n_N}}{(q_N^2 - m_N^2 + i\epsilon)} \times (\text{function of } p\text{'s} \,\&\, q\text{'s})$$

$$\times (2\pi)^4 \delta^4\left( \left(\sum_{i=1}^{k} p_i\right) - \left(\sum_{j=k+1}^{N} q_j\right)\right)$$

$$+ (\text{other contributions}).$$

The properly normalized one particle state $|n_i; \vec{p}_i\rangle$ is extracted from $\mathcal{O}_{n_i}(x^\mu)|\Omega\rangle / \sqrt{Z_{n_i}}$

$$\Rightarrow \boxed{ {}^{\text{out}}\langle (n_{k+1}; \vec{q}_{k+1}) \cdots (n_N; \vec{q}_N) | (n_1; \vec{p}_1) \cdots (n_k. \vec{p}_k) \rangle^{\text{in}} = \left(\prod_{i=1}^{N} \sqrt{Z_{n_i}}\right) \times (2\pi)^4 \delta^4\left( (\Sigma_i p_i) - (\Sigma_j'' q_j) \right) \times (\text{fcn of } p\text{'s and } q\text{'s}) }$$

(This formula makes sense only in a QFT model where some $\langle\Omega|T\{\mathcal{O}(x)\,\mathcal{O}'(0)\}|\Omega\rangle$'s have poles in the spectral representation (= particle picture exists).)

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## §3.2.3 Perturbative Computation of Time-ordered Correlators.

(S-matrix) $\xleftarrow{\text{LSZ formula}}$ (time-ordered correlators) $\longleftarrow$ how can we compute?

$H$ : Hamiltonian of the full theory.

$H_0$ : " : in the free limit. (interaction coupling const $\to 0$)

$H = H_0 + V$

Interaction picture operator : of a Schrödinger picture operator $\mathcal{O}_S(\vec{x})$

$$\boxed{\begin{aligned} \mathcal{O}(\vec{x},t) &= e^{-i\int_t^{t_*} dt' H}\, \mathcal{O}_S(\vec{x})\, e^{-i\int_{t_*}^{t} dt' H} \quad (\text{@ reference time } t_*) \\ \mathcal{O}_I(\vec{x},t) &:= \left(e^{-i\int_t^{t_*} dt' H_0}\right) \mathcal{O}_S(\vec{x}) \left(e^{-i\int_{t_*}^{t} dt' H_0}\right) \end{aligned}}$$

Example : weakly interacting fermion model.

$\psi(\vec{x}) = \Sigma_n \psi_n(\vec{x})\, a_n$ (one particle states $|n\rangle = a_n^\dagger|0\rangle$ in the coupling $\to 0$ limit)
Schrödinger picture op.

$\psi_I(\vec{x},t) = \Sigma_n \psi_n(\vec{x})\, e^{-iE_n(t-t_*)}\, a_n$

$\psi(\vec{x},t)$ is much more complicated when coupling $\neq 0$.

**Step 1**

$$\boxed{\langle \Omega | T\{\mathcal{O}_1(x_1) \cdots \mathcal{O}_N(x_N)\} | \Omega \rangle = \frac{\langle 0 | T\{\mathcal{O}_{1,I}(x_1) \cdots \mathcal{O}_{N,I}(x_N) \exp[-i\int dt'\, V_I(t')]\} | 0 \rangle}{\langle 0 | T\{\exp[-i\int dt'\, V_I(t')]\} | 0 \rangle}}$$

$|\Omega\rangle$ : vacuum state (ground state) of the interacting theory

$|0\rangle$ : " " in the free limit.

$\Rightarrow$ RHS: $\mathcal{O}_I$'s, $V_I$'s : written in terms of the creat'n / ann'tn operators in the free limit theory $\Rightarrow$ computat'n procedure in Step.

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## Derivation of "Step 1"

harmless insertion

- $$\mathcal{O}(t) \overset{(\mathrm{def})}{=} \left(e^{-i\int_t^{*} H}\right) \mathcal{O}_S \left(e^{-i\int_*^{t} H}\right) = e^{-i\int_t^{*} H} \left(e^{-i\int_*^{t} H_0}\, e^{-i\int_t^{*} H_0}\right) \mathcal{O}_S \left(e^{-i\int_*^{t} H_0}\, e^{-i\int_t^{*} H_0}\right) e^{-i\int_*^{t} H}$$
$$= e^{-i\int_t^{*} H}\, e^{-i\int_*^{t} H_0}\, \mathcal{O}_I(t)\, e^{-i\int_t^{*} H_0}\, e^{-i\int_*^{t} H}.$$

- for $t_1 > t_2$

$$\mathcal{O}_1(t_1)\, \mathcal{O}_2(t_2) = e^{-i\int_{t_1}^{*} H}\, e^{-i\int_*^{t_1} H_0}\, \mathcal{O}_{1,I}(t_1)\, e^{-i\int_{t_1}^{*} H_0}\, e^{-i\int_*^{t_1} H}\; e^{-i\int_{t_2}^{*} H}\, e^{-i\int_*^{t_2} H_0}\, \mathcal{O}_{2,I}(t_2)\, e^{-i\int_{t_2}^{*} H_0}\, e^{-i\int_*^{t_2} H}$$
$$= e^{-i\int_{t_1}^{*} H}\, e^{-i\int_*^{t_1} H_0}\, \mathcal{O}_{1,I}(t_1)\, e^{-i\int_{t_1}^{*} H_0}\, e^{-i\int_{t_2}^{t_1} H}\, e^{-i\int_*^{t_2} H_0}\, \mathcal{O}_{2,I}(t_2)\, e^{-i\int_{t_2}^{*} H_0}\, e^{-i\int_*^{t_2} H}.$$

- $$e^{-i\int_{t_f}^{*} H_0}\, e^{-i\int_{t_i}^{t_f} H}\, e^{-i\int_*^{t_i} H_0} = e^{-i\int_{t_i}^{t_f} V_I}$$

- for $t_1 > t_2$, choose $T_+ \gg t_1 > t_2 \gg T_-$

$$\mathcal{O}(t_1)\mathcal{O}(t_2) = \left(e^{-i\int_{T_+}^{*} H}\, e^{-i\int_{t_1}^{T_+} H}\right) e^{-i\int_*^{t_1} H_0}\, \mathcal{O}_{1,I}(t_1) \left(e^{-i\int_{t_2}^{t_1} V_I}\right) \mathcal{O}_{2,I}(t_2)\, e^{-i\int_{t_2}^{*} H_0} \left(e^{-i\int_{T_-}^{t_2} H}\, e^{-i\int_*^{T_-} H}\right)$$
$$= e^{-i\int_{T_+}^{*} H} \underbrace{\left(e^{-i\int_*^{T_+} H_0}\, e^{-i\int_{T_+}^{*} H_0}\right)}_{\text{harmless}} e^{-i\int_{t_1}^{T_+} H}\, e^{-i\int_*^{t_1} H_0}\, \mathcal{O}_{1,I}(t_1) \left(e^{-i\int_{t_2}^{t_1} V_I}\right) \mathcal{O}_{2,I}(t_2)\, e^{-i\int_{t_2}^{*} H_0}\, e^{-i\int_{T_-}^{t_2} H} \left(e^{-i\int_*^{T_-} H_0}\, e^{-i\int_{T_-}^{*} H_0}\right) e^{-i\int_*^{T_-} H}$$
$$= e^{-i\int_{T_+}^{*} H}\, e^{-i\int_*^{T_+} H_0} \left(e^{-i\int_{t_1}^{T_+} V_I}\, \mathcal{O}_{1,I}(t_1)\, e^{-i\int_{t_2}^{t_1} V_I}\, \mathcal{O}_{2,I}(t_2)\right) e^{-i\int_{T_-}^{*} H_0}\, e^{-i\int_*^{T_-} H}.$$

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So,

$$\langle\Omega|T\{O_1(t_1)\,O_2(t_2)\}|\Omega\rangle = \langle\Omega|\,e^{-i\int_{T_+}^{*}H}\,e^{-i\int_{*}^{T_+}H_0}\,T\Big\{O_{1,I}(t_1)\,O_{2,I}(t_2)\,\exp\Big[-i\int_{T_-}^{T_+}dt'\,V_I(t')\Big]\Big\}\,e^{-i\int_{T_-}^{*}H_0}\,e^{-i\int_{*}^{T_-}H}|\Omega\rangle.$$

- $$e^{-i\int_{T_-}^{*}H}\,e^{-i\int_{*}^{T_-}H_0}|0\rangle \propto |\Omega\rangle =: Z|\Omega\rangle$$

$$\longrightarrow\ e^{-i\int_{T_-}^{*}H_0}\,e^{-i\int_{*}^{T_-}H}|\Omega\rangle = Z^{-1}|0\rangle.$$

$$\langle 0|\,e^{-i\int_{T_+}^{*}H_0}\,e^{-i\int_{*}^{T_+}H} \propto \langle\Omega| =: Z^{\dagger}\langle\Omega|$$

$$\longrightarrow\ \langle\Omega|\,e^{-i\int_{T_+}^{*}H}\,e^{-i\int_{*}^{T_+}H_0} = \langle 0|\,Z^{\dagger -1}.$$

$$|Z|^2 = |Z|^2\langle\Omega|\Omega\rangle = \langle 0|\,e^{-i\int_{T_+}^{*}H_0}\,e^{-i\int_{T_-}^{T_+}H}\,e^{-i\int_{*}^{T_-}H_0}|0\rangle$$

$$= \langle 0|\,e^{-i\int_{T_-}^{T_+}V_I}\,|0\rangle.$$

- Combining all above together, we obtain

$$\langle\Omega|T\{O_1(t_1)\,O_2(t_2)\}|\Omega\rangle = \frac{1}{|Z|^2}\langle 0|T\{O_{1,I}(t_1)\,O_{2,I}(t_2)\,e^{-i\int_{T_-}^{T_+}V_I}\}|0\rangle$$

$$= \frac{\langle 0|T\{O_{1,I}(t_1)\,O_{2,I}(t_2)\,e^{-i\int_{T_-}^{T_+}dt'\,V_I(t')}\}|0\rangle}{\langle 0|T\{\exp[-i\int_{T_-}^{T_+}dt'\,V_I(t')]\}|0\rangle}$$

(Step. 1)

$\boxed{\text{Step 2 : Feynman rule}}$ (propagators & vertex factors)

Interaction-picture elementary field operators

◎ real scalar field $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 + \cdots$

free part canonical quantization

relativistic normalization

$$\Rightarrow [a_{\vec p}, a_{\vec q}^\dagger] = (2\pi)^3 \delta^3(\vec p - \vec q) \qquad |\vec p\rangle = a_{\vec p}^\dagger |0\rangle \sqrt{2E_{\vec p}}$$

$$\phi_I(x^\mu) = \int \frac{d^3p}{(2\pi)^3} \left( e^{-ip\cdot x} a_{\vec p} + e^{ip\cdot x} a_{\vec p}^\dagger \right) \frac{1}{\sqrt{2E_{\vec p}}}$$

◎ complex scalar field $\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 |\phi|^2 + \cdots$

$$\Rightarrow [a_{\vec p}, a_{\vec q}^\dagger] = [b_{\vec p}, b_{\vec q}^\dagger] = (2\pi)^3 \delta^3(\vec p - \vec q)$$

$$\phi_I(x) = \int \frac{d^3p}{(2\pi)^3} \left( e^{-ip\cdot x} a_{\vec p} + e^{ip\cdot x} b_{\vec p}^\dagger \right) \frac{1}{\sqrt{2E_{\vec p}}}$$

$$\phi_I^*(x) = \int \frac{d^3p}{(2\pi)^3} \left( e^{-ip\cdot x} b_{\vec p} + e^{ip\cdot x} a_{\vec p}^\dagger \right) \frac{1}{\sqrt{2E_{\vec p}}}$$

straight forward computation (bring ann. op's to the right to $|0\rangle$, creat'n op's to the left to $\langle 0|$, using the comm. relat'n.)

real scalar

$$\boxed{\langle 0| T\{ \phi_I(x) \phi_I(y) \} |0\rangle = \int \frac{d^4p}{(2\pi)^4} \frac{i e^{-ip\cdot(x-y)}}{(p^2 - m^2 + i\varepsilon)}}$$

complex scalar

$$\langle 0| T\{ \phi_I(x) \phi_I^*(y) \} |0\rangle = \int \frac{d^4p}{(2\pi)^4} \frac{i e^{-ip\cdot(x-y)}}{(p^2 - m^2 + i\varepsilon)}$$

$$\langle 0| T\{ \phi_I(x) \phi_I(y) \} |0\rangle = 0$$

$$\langle 0| T\{ \phi_I^*(x) \phi_I^*(y) \} |0\rangle = 0$$

◎ 4-component spinor field $\mathcal{L} = \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi + \cdots\cdots$

$$\Rightarrow \{a_{\vec{p},s}, a^\dagger_{\vec{q},r}\} = (2\pi)^3 \cdot \delta^3(\vec{p}-\vec{q}) = \{b_{\vec{p},s}, b^\dagger_{\vec{q},r}\}$$

$$|e^-; \vec{p},s\rangle = a^\dagger_{\vec{p},s}|0\rangle\sqrt{2E_{\vec{p}}} \qquad |e^+; \vec{p},s\rangle = b^\dagger_{\vec{p},s}|0\rangle\sqrt{2E_{\vec{p}}}$$

$$\Psi_I(x) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left(\sum_{s=\uparrow,\downarrow} u_s(\vec{p})\, a_{\vec{p},s}\, e^{-ip\cdot x} + \sum_{r=\uparrow,\downarrow} v_s(\vec{p})\, b^\dagger_{\vec{p},s}\, e^{ip\cdot x}\right)$$

$$\bar{\Psi}_I(x) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left(\sum_s \bar{u}_s(\vec{p})\, a^\dagger_{\vec{p},s}\, e^{ip\cdot x} + \sum_r \bar{v}_s(\vec{p})\, b_{\vec{p},s}\, e^{-ip\cdot x}\right)$$

$$\Rightarrow \boxed{\langle 0|T\{\Psi_I(x)\bar{\Psi}_I(y)\}|0\rangle = \int\frac{d^4p}{(2\pi)^4}\frac{i(\not{p}+m)\, e^{-ip\cdot(x-y)}}{(p^2-m^2+i\epsilon)}}$$

The 4-component "polarization"s :

used to derive that.

$$u_s(\vec{p}) = \begin{pmatrix}\sqrt{p\cdot\sigma}\ \xi_s \\ \sqrt{p\cdot\bar{\sigma}}\ \xi_s\end{pmatrix} \qquad v_s(\vec{p}) = \begin{pmatrix}\sqrt{p\cdot\sigma}\ \eta_s \\ -\sqrt{p\cdot\bar{\sigma}}\ \eta_s\end{pmatrix}$$

$$\gamma^\mu = \begin{pmatrix}0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0\end{pmatrix} \quad \sigma^{\mu=0,i} = (\mathbb{1}, \tau^i)_{2\times2}, \quad \bar{\sigma}^{\mu=0,i} = (\mathbb{1}, -\tau^i)_{2\times2}$$

$\xi_r^\dagger \xi_s = \delta_{r,s}$

$\eta_r^\dagger \eta_s = \delta_{r,s}$

$$\Rightarrow \sum_{s=\uparrow,\downarrow} u_s(\vec{p})\bar{u}_s(\vec{p}) = (\not{p}+m)$$

$$\sum_{r=\uparrow,\downarrow} v_r(\vec{p})\bar{v}_r(\vec{p}) = (\not{p}-m)$$

$\sum_s \xi_s\xi_s^\dagger = \mathbb{1}_{2\times2}$

$\sum_r \eta_r\eta_r^\dagger = \mathbb{1}_{2\times2}$

also
$$u_r^\dagger(\vec{p})\, u_s(\vec{p}) = 2E_{\vec{p}}\ \xi_r^\dagger\xi_s = v_r^\dagger(\vec{p})\, v_s(\vec{p})$$

$$\bar{u}_r(\vec{p})\, u_s(\vec{p}) = 2m\ \xi_r^\dagger\xi_s = -\bar{v}_r(\vec{p})\, v_s(\vec{p})$$

(See Peskin Schroeder §3 for more info.)

◎ 2-component spinor field $\mathcal{L} = \psi^\dagger i\bar{\sigma}^\mu(\partial_\mu\psi) - \frac{m}{2}\psi^T\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\psi - \frac{m}{2}\psi^\dagger\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\psi^* + \cdots$

$$\psi_I(x) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\sum_{s=\uparrow\downarrow}\left(\sqrt{p\cdot\sigma}\ \xi_s\, a_{\vec{p},s}\, e^{-ip\cdot x} + \sqrt{p\cdot\sigma}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\xi_s^*\, a^\dagger_{\vec{p},s}\, e^{ip\cdot x}\right)$$

$$\boxed{\begin{aligned}\langle 0|T\{\psi_I(x)\psi_I^\dagger(y)\}|0\rangle &= \int\frac{d^4p}{(2\pi)^4}\frac{i(p\cdot\sigma)\, e^{-ip\cdot(x-y)}}{(p^2-m^2+i\epsilon)} \\ \langle 0|T\{\psi_I(x)\psi_I^T(y)\}|0\rangle &= \int\frac{d^4p}{(2\pi)^4}\frac{im\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} e^{-ip\cdot(x-y)}}{(p^2-m^2+i\epsilon)}\end{aligned}}$$

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## Vertex factors

To compute $\langle 0|T\{ \cdots \exp[-i\int dt' V_I(t')] \}|0\rangle$,

use the Taylor series expansion of $\exp[-i\int V_I]$.

Example : QED.

$\mathcal{L} \supset \bar{\Psi}_f \left(i\gamma^\mu(\partial_\mu + ieQ_f A_\mu) - m_f\right)\Psi_f$ f: fermion (e.g. e, μ, τ, u, d, etc.)

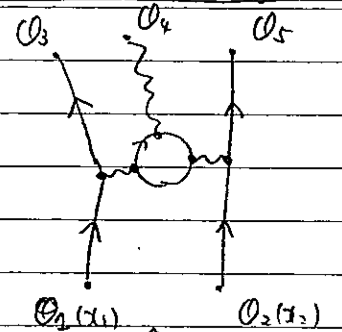
$\Rightarrow +\int d^3x \left(\bar{\Psi}_f \gamma^\mu A_\mu \Psi_f\right) eQ_f$ is in $V$ (the interaction part).

$e>0$ ($Q_e=-1$, $Q_u=+2/3$, $Q_d=-1/3$)

The first order term in the expansion

has a factor $-ieQ_f\, d^4x\, \gamma^\mu$ besides interaction-picture operators.

## Feynman diagram




use $\frac{1}{5!}\left(-i\int dt' V_I\right)^5$

use $-i\int dt' V_I(t')$ $\Rightarrow$ include the vertex factor — draw a vertex

exploit comm. / anti-comm relat'n for a pair of int'n-picture operators $\Rightarrow$ only a $\mathbb{C}$-valued function is left. (propagator) (not an operator any more) — draw a line.

Step 3

$\#$(momentum integrat'n) = $\#$(internal lines) $-$ {$\#$(vertex) $-$ $\#$(connected components of the graph)}

momentum conservation

5 5 1

= ($\#$ loops) in the graph 1.

[topology: $\#$(1-dim objects) $-$ $\#$(0-dim objects) = $\#$(loops) $-$ $\#$(connct'd compnts)

Higher loop contributns often involve higher power of coupling constants $\Rightarrow$ smaller (if weak coupling)

## §2.4 Other quantities of interest in QFT

Time-ordered product correlation functions
(and scattering amplitudes derived from them)
are not the only class of observables in QFT's.

Two other classes of interest.

- In-In formalism / real time formalism in thermal field theory / Schwinger-Keldysh formalism (known in many different names).

examples: $\langle \text{init. state} | T\{\mathcal{O}_1(x_1)\, \mathcal{O}_2(x_2) \cdots \mathcal{O}_n(x_n)\} | \text{init. state} \rangle$

The initial state $|\text{init. state}\rangle$ is often the ground state or thermal ensemble, and interactions (backgrounds) are turned on later

✓ conductance (current-current correlation) in a time-dep. background field

✓ decay / scattering of particles in a thermal plasma.

✓ growth (time evolution) of inflation field fluctuations in the background of metric of expanding universe

The time coordinate starts at $t = t_*$ for the initial state. It then goes up (operators are sorted out that way), and the comes back to $t_*$ in this class of observables.

- Out-of-time-ordered correlation functions.

example $\langle \Omega | \, \left| [A(t,\vec{x}), B(0,\vec{y})] \right|^2 |\Omega\rangle$ — (*). vanishes @ $t=0$ $(\vec{x} \neq \vec{y})$. How fast does it grow?

If $A$ and $B$ are unitary operators

$$(*) = 2 - \left( \langle \Omega | B^\dagger A^\dagger B A | \Omega \rangle + \text{c.c.} \right).$$

There are experimental ways to measure such (*)'s.