§4 ~~§3~~ Scattering Processes at the Leading Order

§4.1 ~~§3.1~~ Vector field propagators.

gauge symmetry = redundant description.

$A_\mu(x,t)$ and $A'_\mu := A_\mu - (\partial_\mu \chi(x,t))$ are physically equivalent.

Choose a gauge

example: Coulomb gauge $A_0 = 0$ and $\vec{\nabla}\cdot\vec{A} = 0$.

when the photon is not coupled to matter.

Mode decomposition + canonical quantization.

$$\Rightarrow \begin{cases} \vec{A}(\vec{x},t) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_{r=1,2} \left(\vec{\varepsilon}_r(\vec{p})\, a_{\vec{p},r}\, \underline{\underline{e^{-ip\cdot x}}} + \vec{\varepsilon}_r(\vec{p})\, a^\dagger_{\vec{p},r}\, \underline{\underline{e^{ip\cdot x}}} \right) \\ A_0(\vec{x},t) = 0. \end{cases}$$

$\vec{p}\cdot\vec{\varepsilon}_r(\vec{p}) = 0$. (transverse polarization)

This results in

$$G_{ij}(x,y) := \langle 0 | T\{A_i(x) A_j(y)\} | 0 \rangle$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{i e^{-ip\cdot(x-y)}}{p^2 + i\varepsilon} \left[\sum_r \vec{\varepsilon}_r(\vec{p}) \otimes \vec{\varepsilon}_r(\vec{p}) \right]_{ij} \rightarrow \left[\mathbb{1}_{ij} - \frac{\vec{p}_i \otimes \vec{p}_j}{|\vec{p}|^2} \right].$$

But this Green function does not satisfy the EOM.

$$\left[\partial^2 \delta_\mu{}^\kappa - \partial_\mu \partial^\kappa \right] G_{\kappa\lambda} J^\lambda = J_\mu \ldots \qquad (\divideontimes)$$

Modification A (temporal / axial gauge)

$$\boxed{G_{\mu\nu} = \frac{-i}{p^2 + i\varepsilon} \left[\eta_{\mu\nu} - \frac{(p_\mu n_\nu + n_\mu p_\nu)}{(p\cdot n)} + \frac{p_\mu p_\nu\, n^2}{(p\cdot n)^2} \right]}$$ for some n_μ.

For $n^\mu = (1,\vec{0})$ $\quad n^\mu G_{\mu\nu} = 0$, $\quad p^i G_{i\nu} J^\nu = \frac{i\rho(p)}{p^0}$, $(\divideontimes)$

are all satisfied. [use $\partial^\mu J_\mu = 0$]

The 3×3 part is $\left[-\mathbb{1} + \frac{\vec{p}\otimes\vec{p}}{(p^0)^2} \right]$

(Modification B (Coulomb gauge)

replace by $\left[\eta_{\mu\nu} - \frac{(p_\mu n_\nu + n_\mu p_\nu)(n\cdot p)}{(p\cdot n)^2 - p^2 n^2} + \frac{p_\mu p_\nu\, n^2}{(p\cdot n)^2 - p^2 n^2} \right]$) cf. (hw. V-1)

The Coulomb gauge (or axial gauge, temporal gauge) propagator is rarely use for practical computations in QFT.

More convenient choice is

$$\boxed{G_{\mu\nu} = \int \frac{d^4p}{(2\pi)^4} \frac{i e^{-ip\cdot(x-y)}}{p^2+i\varepsilon} \left[\eta_{\mu\nu} + (\xi-1)\frac{p_\mu p_\nu}{p^2}\right].}$$

- $\xi = 1$: Feynman gauge.
- $\xi = 0$: Landau gauge
- $\xi = \infty$: unitary gauge.

(For derivation, see Peskin-Schroeder §9.)

supplementary notes.

Maxwell equation

$$\begin{cases} \mathrm{div}\,\vec{E} = \rho/\varepsilon_0 \\ \mathrm{rot}\,\vec{B} - \dfrac{1}{c^2}\dfrac{\partial \vec{E}}{\partial t} = \mu_0 \vec{j} \end{cases} \qquad \begin{cases} \mathrm{div}\,\vec{B} = 0 \\ \mathrm{rot}\,\vec{E} + \dfrac{\partial \vec{B}}{\partial t} = \vec{0} \end{cases}$$

$$\vec{f} = eQ(\vec{E} + \vec{v}\times\vec{B})$$

$$\varepsilon_0\mu_0 = 1/c^2.$$

$$\varphi = \frac{1}{4\pi\varepsilon_0}\frac{eQ}{r}$$

Physics always comes in a combination of (charge) × (field).

So it is OK. to rescale $\left[\begin{array}{l}(\text{charge}) \to (\text{charge})' = (\text{charge})\times\lambda. \\ (\text{field}) \to (\text{field})' = (\text{field})\times\lambda^{-1}.\end{array}\right]$.

Using this rescaling, we can set.

	cgs-esu	cgs-esu rational	cgs-emu	cgs-emu rational
ε_0	$1/4\pi$	1	$1/(4\pi c^2)$	$1/c^2$
μ_0	$4\pi/c^2$	$1/c^2$	4π	1

$$(\text{charge})_{\text{in esu}} = (\text{charge})_{\text{in emu}} \times (c_{\text{value}}).$$

$$(\text{field})_{\text{in esu}} = (\text{field})_{\text{in emu}} / (c_{\text{value}})$$

cgs-Gauss. unit system

$$A^\mu = (\varphi_{esu},\ c\cdot\vec{A}_{esu}) = \left(\frac{1}{c}\varphi_{emu},\ \vec{A}_{emu}\right) \quad \text{and} \quad J^\mu = \left(\rho_{esu},\ \frac{1}{c}\vec{j}_{esu}\right) = (c\,\rho_{emu},\ \vec{j}_{emu})$$

$$\boxed{\begin{array}{c} \partial_\nu F^{\nu\mu} = 4\pi J^\mu \\ A^0 = \dfrac{eQ}{r} \\ \mathcal{L} = -\dfrac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} - eA_\mu J^\mu \\ \mathcal{H}_{EM} = \dfrac{1}{8\pi}(\vec{E}^2 + \vec{B}^2) \end{array}}$$

"rational" version

$$\boxed{\begin{array}{c} \partial_\nu F^{\nu\mu} = J^\mu. \\ A^0 = \dfrac{1}{4\pi}\dfrac{eQ}{r} \\ \mathcal{L} = -\dfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - eA_\mu J^\mu \\ \mathcal{H}_{EM} = \dfrac{1}{2}(\vec{E}^2 + \vec{B}^2) \end{array}}$$

$$(\text{charge})_{\text{in non-rat}} = (\text{charge})_{\text{in rat.unit.}} / \sqrt{4\pi}$$

$$(\text{field})_{\text{in non-rat}} = (\text{field})_{\text{in rat.unit}} \times \sqrt{4\pi}$$

- Fine str. constant

$$\alpha := \left(\frac{e^2}{4\pi\varepsilon_0}\right) \Rightarrow \frac{(e_{esu})^2}{4\pi\cdot(1/4\pi)} = \frac{(e_{Gauss})^2}{4\pi\cdot(1/4\pi)} = \frac{(e_{emu})^2}{4\pi\cdot(1/4\pi c^2)} = \boxed{\frac{(e_{Gauss\text{-}rat})^2}{4\pi\cdot 1}}$$

$\alpha := e^2/4\pi$ in the lecture note.

"e_{unit}" is the value of (positive) unit charge in a given unit system

- $J^\mu = \cancel{Q_i\bar{\psi}_i(i\gamma^\mu\partial_\mu - m)\psi_i}$ $\Sigma_i Q_i(\bar{\psi}_i\gamma^\mu\psi_i)$ means that $\Delta\mathcal{L} = \bar{\psi}_i\{i\gamma^\mu(\partial_\mu + ieQ_iA_\mu) - m_i\}\psi_i$.

sign.

§~~3.2~~ 4.2. $e^+ + e^- \to \mu^+ + \mu^-$ cross section

We use a Lagrangian $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar\psi(i\gamma^\mu D_\mu - m)\psi$.

$$D_\mu = (\partial_\mu \cancel{-} + ieQA_\mu)$$

$$\begin{pmatrix} e>0. & \\ Q:\ \text{charge.} & (Q_e=-1) \end{pmatrix}$$

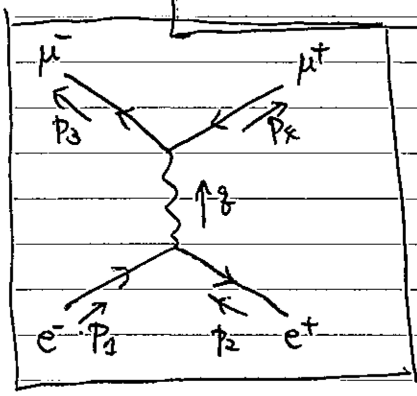
Consider.

$$(*) = \langle 0|T\Big\{\psi_{m.c}(-p_3)\,\bar\psi_{m.d}(-p_4)\,\psi_{e.b}(p_2)\,\bar\psi_{e.a}(p_1)\int d^4x(-ie)\{Q_m(\bar\psi_m\gamma^\mu\psi_m)+Q_e(\bar\psi_e\gamma^\mu\psi_e)\}A_\mu$$
$$\int d^4y(-ie)\{Q_m(\bar\psi_m\gamma^\nu\psi_m)+Q_e(\bar\psi_e\gamma^\nu\psi_e)\}A_\nu \times\frac{1}{2!}\Big\}|0\rangle.$$

$(p_1^{\mu=0},\ p_2^{\mu=0},\ p_3^{\mu=0},\ p_4^{\mu=0} > 0.)$

$\sqcap$ (contraction) : pairing of annihilation and creation operators

$$\Rightarrow \frac{2}{2}\times\frac{i(\not p_3+m_m)_{ef}}{p_3^2-m_m^2+i\varepsilon}(-ieQ_m)[\gamma^\mu]_{fg}\frac{i(-\not p_4+m_m)_{gd}}{p_4^2-m_m^2+i\varepsilon}\times\left(\frac{-i\eta_{\mu\nu}}{q^2+i\varepsilon}\right)$$
$$\times\frac{i(-\not p_2+m_e)_{bh}}{p_2^2-m_e^2+i\varepsilon}(-ieQ_e)[\gamma^\nu]_{hj}\frac{i(\not p_1+m_e)_{ja}}{p_1^2-m_e^2+i\varepsilon}$$
$$\times(2\pi)^4\delta^4(p_1+p_2-q)\times(2\pi)^4\delta^4(q-p_3-p_4)\quad\frac{d^4q}{(2\pi)^4}$$



$$\mathrm{Res}((*)) = [\bar u_r(\vec p_3)\gamma^\mu v_s(\vec p_4)][\bar v_{s'}(\vec p_2)\gamma^\nu u_{r'}(\vec p_1)]\frac{\eta_{\mu\nu}}{(p_1+p_2)^2+i\varepsilon}$$
$$\times\ i(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)\times(e^2Q_eQ_\mu)$$
$$=(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)\ i\mathcal{M}(e^+_{s'}e^-_{r'}\to\mu^+_s\mu^-_r).$$

(LSZ formula)

Now, we hope to compute $|\mathcal{M}|^2$ (and then the cross section).

But the expression is a mess, when we retain m_e, m_μ etc. r, s, r', s'.

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§4.2.1 High-energy limit and threshold limit (unpolarized case)

The expression of $|\mathcal{M}|^2$ simplifies when we think of

$$\longrightarrow \sum_{r,s} \left|\mathcal{M}\left(e^-_{r'} + e^+_{s'} \to \mu^-_r + \mu^+_s\right)\right|^2$$

ignore the spin of μ^+, μ^- in the final state.

$$\longrightarrow \frac{1}{2}\sum_{r'=1}^{2} \frac{1}{2}\sum_{s'=1}^{2} \sum_{r,s=1}^{2} \left|\mathcal{M}\left(e^-_{r'} + e^+_{s'} \to \mu^-_r + \mu^+_s\right)\right|^2$$

initial state e^+, e^- are not polarized.

Now

$$\overline{\Sigma|\mathcal{M}|^2} = \frac{1}{4}\sum_{r',s',r,s}\left(\frac{e^2 Q_e Q_\mu}{(p_1+p_2)^2}\right)^2 \times \left[\bar{u}_r(\vec{p}_3)\gamma^\mu v_s(\vec{p}_4)\right]\left[\bar{v}_s(\vec{p}_4)\gamma^\kappa u_r(\vec{p}_3)\right]$$

$$\times \left[\bar{v}_{s'}(\vec{p}_2)\gamma_\mu u_{r'}(\vec{p}_1)\right]\left[\bar{u}_{r'}(\vec{p}_1)\gamma_\kappa v_{s'}(\vec{p}_2)\right]$$

$$= \frac{1}{4}\left(\frac{e^2 Q_e Q_\mu}{(p_1+p_2)^2}\right)^2 \times \underset{4\times4}{\mathrm{Tr}}\left[\gamma^\mu(\not{p}_4 - m_\mu)\gamma^\kappa(\not{p}_3 + m_\mu)\right] \times \underset{4\times4}{\mathrm{Tr}}\left[\gamma_\mu(\not{p}_1 + m_e)\gamma_\kappa(\not{p}_2 - m_e)\right]$$

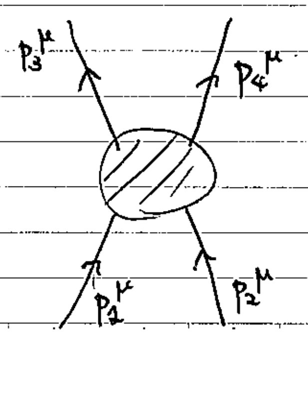
(see. suppl. notes.)

$$= \frac{1}{4}\left(\frac{e^2 Q_e Q_\mu}{(p_1+p_2)^2}\right)^2 \times \left\{-m_\mu^2\eta^{\mu\kappa} - (p_3\cdot p_4)\eta^{\mu\kappa} + p_4^\mu p_3^\kappa + p_4^\kappa p_3^\mu\right\}\times 4$$

$$\times\left\{-m_e^2\eta_{\mu\kappa} - (p_1\cdot p_2)\eta_{\mu\kappa} + p_{1\mu}p_{2\kappa} + p_{1\kappa}p_{2\mu}\right\}\times 4$$

$$= 4\left(\frac{e^2 Q_e Q_\mu}{(p_1+p_2)^2}\right)^2 \times \Big[\left\{(p_3\cdot p_4) + m_\mu^2\right\}\left\{(p_1\cdot p_2) + m_e^2\right\}4$$
$$+\left\{(p_1\cdot p_4)(p_2\cdot p_3) + (p_4\cdot p_2)(p_1\cdot p_3)\right\}\times 2$$
$$-\left[(p_1\cdot p_2) + m_e^2\right](p_3\cdot p_4)\,2 - \left[(p_3\cdot p_4) + m_\mu^2\right](p_1\cdot p_2)\cdot 2\Big].$$

It is conventional to use the following variables (Mandelstam variables) in 2-body $\to$ 2-body scattering:



$$s := (p_1+p_2)^2$$
$$t := (p_1-p_3)^2$$
$$u := (p_1-p_4)^2.$$

Due to the momentum conservation $(p_1+p_2-p_3-p_4)^\mu = 0$, there is a relation

$$s+t+u = m_1^2+m_2^2+m_3^2+m_4^2.$$

(Q: verify this relation.)

No
Date

$\boxed{\text{Supplementary notes}}$

$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$: the definition of gamma matrices.

We can use this anti-commutation relation to derive the followings.

First,

$$\mathrm{tr}_{4\times4}[\gamma^\mu\gamma^\nu] \underset{\substack{\uparrow \\ \text{cyclic rotation within a trace.}}}{=} \frac{1}{2}\left(\mathrm{tr}[\gamma^\mu\gamma^\nu] + \mathrm{tr}[\gamma^\nu\gamma^\mu]\right) = \frac{1}{2}\mathrm{tr}[\{\gamma^\mu, \gamma^\nu\}] = \frac{2}{2}\mathrm{tr}[\mathbb{1}_{4\times4}]\eta^{\mu\nu}$$
$$= 4\eta^{\mu\nu}.$$

Secondly,

$$\mathrm{tr}_{4\times4}[\gamma^\mu\gamma^\nu\gamma^\kappa\gamma^\lambda] = \mathrm{tr}_{4\times4}[\{\gamma^\mu,\gamma^\nu\}\gamma^\kappa\gamma^\lambda] - \mathrm{tr}_{4\times4}[\gamma^\nu\gamma^\mu\gamma^\kappa\gamma^\lambda]$$
$$= 2\eta^{\mu\nu}\mathrm{tr}[\gamma^\kappa\gamma^\lambda] - \mathrm{tr}[\gamma^\nu\{\gamma^\mu,\gamma^\kappa\}\gamma^\lambda] + \mathrm{tr}[\gamma^\nu\gamma^\kappa\gamma^\mu\gamma^\lambda]$$
$$= 2\eta^{\mu\nu}\mathrm{tr}[\gamma^\kappa\gamma^\lambda] - 2\eta^{\mu\kappa}\mathrm{tr}[\gamma^\nu\gamma^\lambda] + \mathrm{tr}[\gamma^\nu\gamma^\kappa\{\gamma^\mu,\gamma^\lambda\}]$$
$$- \mathrm{tr}[\gamma^\nu\gamma^\kappa\gamma^\lambda\gamma^\mu]$$
$$= 2\times4\left(\eta^{\mu\nu}\eta^{\kappa\lambda} - \eta^{\mu\kappa}\eta^{\nu\lambda} + \eta^{\mu\lambda}\eta^{\nu\kappa}\right) - \mathrm{tr}[\gamma^\mu\gamma^\nu\gamma^\kappa\gamma^\lambda].$$

Therefore $\mathrm{tr}_{4\times4}[\gamma^\mu\gamma^\nu\gamma^\kappa\gamma^\lambda] = 4\left(\eta^{\mu\nu}\eta^{\kappa\lambda} - \eta^{\mu\kappa}\eta^{\nu\lambda} + \eta^{\mu\lambda}\eta^{\nu\kappa}\right).$

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Using these kinematical variables, the spin-sum/average ME can be rewritten as follows:

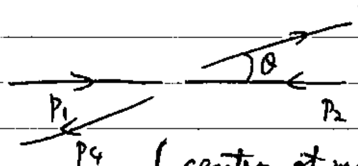
$$\frac{1}{4}\sum_{r',s'}\sum_{r,s}|\mathcal{M}|^2 = 4\left(\frac{e^2 Q_e Q_\mu}{s}\right)^2\left[s^2+\frac{1}{2}\left\{(-u+m_e^2+m_\mu^2)^2+(-t+m_e^2+m_\mu^2)^2\right\} - s\left(\frac{s-2m_\mu^2}{2}\right) - s\left(\frac{s-2m_e^2}{2}\right)\right]$$

$$= 4\left(\frac{e^2 Q_e Q_\mu}{s}\right)^2\left[\frac{u^2+t^2}{2}+(s-u-t)(m_e^2+m_\mu^2)+(m_e^2+m_\mu^2)^2\right].$$

Here are two limits of interest.

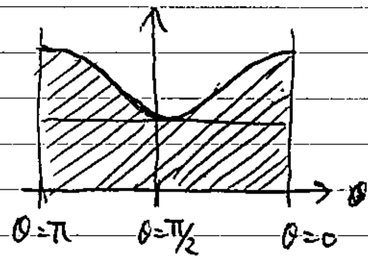
- High energy limit : $(2m_\mu)^2 \ll s$.

$$\Rightarrow \frac{1}{4}\sum_{r',s',r,s}|\mathcal{M}|^2 \cong 2(e^2 Q_e Q_\mu)^2\left(\frac{u^2+t^2}{s^2}\right) = (e^2 Q_e Q_\mu)^2(1+\cos^2\theta)$$



(center of mass frame scattering angle)

$$\begin{cases} t \simeq -s\cdot\sin^2\left(\frac{\theta}{2}\right) \\ u \simeq -s\cos^2\left(\frac{\theta}{2}\right) \end{cases}$$



Using [homework I-1]

$$d\sigma \cong \frac{d(\cos\theta)}{32\pi s}(\beta\simeq 1)\left(\frac{1}{4}\sum_{r's'r.s}|\mathcal{M}|^2\right) \cong \left[d(\cos\theta)(1+\cos^2\theta)\Rightarrow\frac{8}{3}\right]\times\frac{\pi\alpha_e^2}{2s}Q_\mu^2 \Rightarrow \boxed{\frac{4\pi}{3}\frac{\alpha_e^2}{s}(Q_\mu)^2}$$

- Just above the threshold : $s \sim (2m_\mu)^2$ $\quad(\Rightarrow t\sim u\sim -m_\mu^2)$

$$\Rightarrow \frac{1}{4}\sum_{r',s',r,s}|\mathcal{M}|^2 \cong 2(e^2 Q_e Q_\mu)^2 \qquad (\theta\text{-indep.})$$

Using [homework I-1]

$$\int d\sigma \cong \int_{-1}^{+1} d(\cos\theta)\,\frac{\beta}{32\pi s}\left(\frac{1}{4}\sum_{r's'r.s}|\mathcal{M}|^2\right) \cong \boxed{\pi\frac{\alpha_e^2}{s}(Q_\mu)^2\times\beta}$$

$$\boxed{\beta=\frac{|\vec{P}_\mu|}{E_\mu}}$$

$$\boxed{\alpha_e := \frac{e^2}{4\pi}}$$

Thus the weak-electromagnetic interference has been observed unambiguously at PETRA. For the sake of visual display, the combined fit to the angular distributions is shown in fig. 5.13 [5.36].

$\boxed{s\frac{d\sigma}{d\cos\theta(d\varphi)}}$

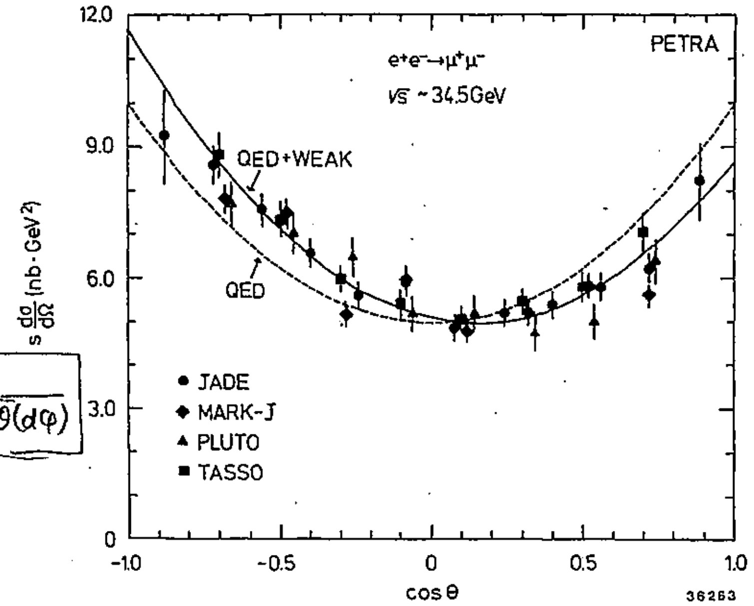


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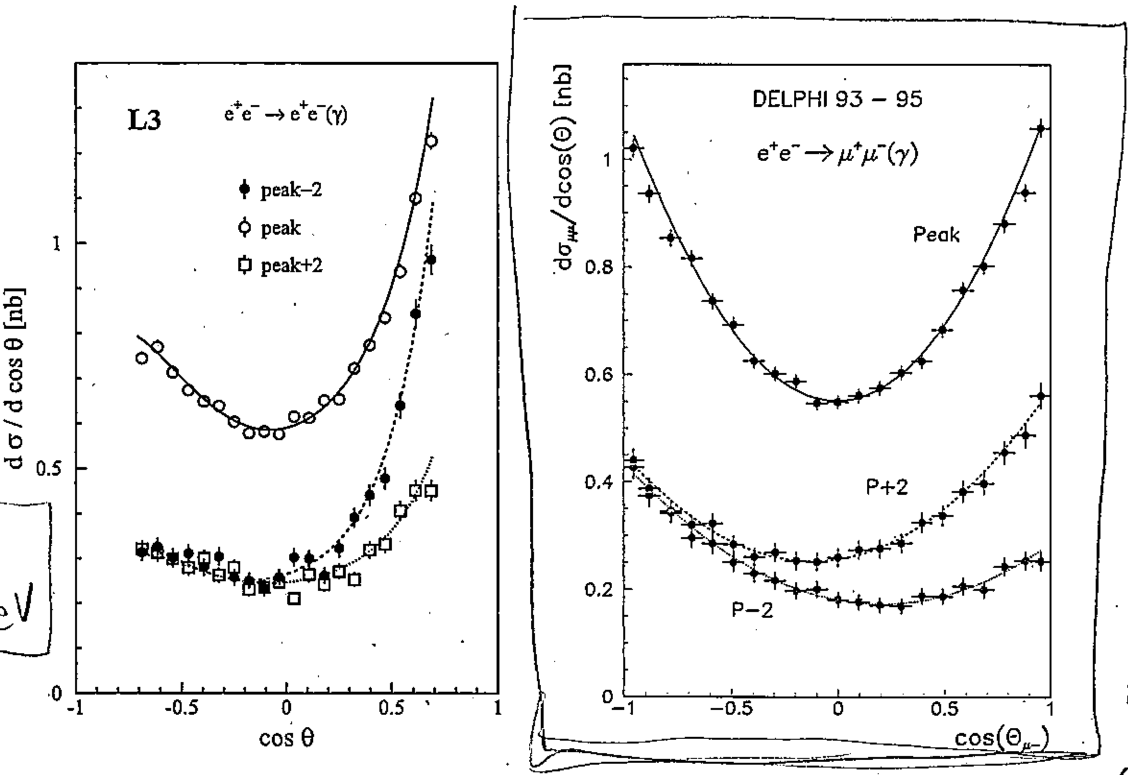
Fig. 5.13. Compilation [5.36] of PETRA high-energy data for the angular distribution of $e^+e^- \to \mu^+\mu^-$ at $\sqrt{s} \sim 34.5$ GeV. The data are corrected for effects α^3. The full curve shows a fit to the data allowing for an asymmetry; the dashed curve is the symmetric QED prediction.

$\boxed{\frac{d\sigma}{d\cos\theta}}$

$\boxed{\sqrt{s}\sim 91\pm2\text{GeV}}$



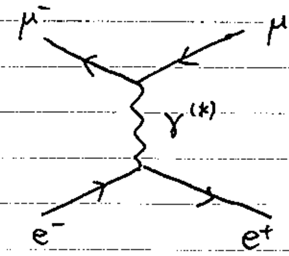
hep-ex/0509008 (LEP review)

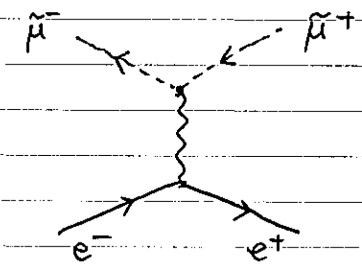
Figure 2.5: Distribution of the production polar angle, $\cos\theta$, for e^+e^- and $\mu^+\mu^-$ events at the three principal energies during the years 1993–1995, measured in the L3 (left) and DELPHI (right) detectors, respectively. The curves show the SM prediction from ALIBABA [52] for e^+e^- and a fit to the data for $\mu^+\mu^-$ assuming the parabolic form of the differential cross-section given in the text.

Compare

$$\left(e^-_{r'} + e^+_{s'} \to \mu^-_{r} + \mu^+_{s} \right) \quad \text{vs} \quad \left(e^-_{r'} + e^+_{s'} \to \tilde{\mu}^- + \tilde{\mu}^+ \right)$$

spin-1/2 spin-0.





High-energy limit

$$\frac{1}{4}\sum_{r's'}\sum_{rs}|\mathcal{M}|^2 \cong \left(e^2 Q(\mu) Q(e)\right)^2 \left(1+\cos^2\theta\right) \quad \text{vs} \quad \frac{1}{4}\sum_{r's'}|\mathcal{M}|^2 \cong \left(e^2 Q(\mu) Q(e)\right)^2 \frac{\sin^2\theta}{2}.$$

Both are dimensionless. (remember: 1st lecture of this course)

only angle dependence remains.

At the threshold

$$\frac{1}{4}\sum_{r's',rs}|\mathcal{M}|^2 \cong \left(e^2 Q(\mu) Q(e)\right)^2 \times 2, \qquad \frac{1}{4}\sum_{r's'}|\mathcal{M}|^2 \cong \left(e^2 Q(\mu) Q(e)\right)^2 \frac{\sin^2\theta}{2} \times \left(\beta(\mu)\right)^2$$

(see homework ~~IV~~ V -2.)

These matrix elements are integrated

$$\text{over} \quad \frac{\beta(\mu)}{32\pi s} \int_{-1}^{+1} d(\cos\theta)$$

Cross sections of 2-body $\to$ 2body hard scattering processes often scale as

$$\sigma \sim \frac{1}{s} \sim \frac{1}{(E_{CM}/\mathrm{GeV})^2} \times \left[\frac{(\hbar c)^2}{(\mathrm{GeV})^2} \sim 4\times 10^{-4}\,\mathrm{bn} \right]$$

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particle data group web page.

52.3 σ and R in e^+e^- Collisions

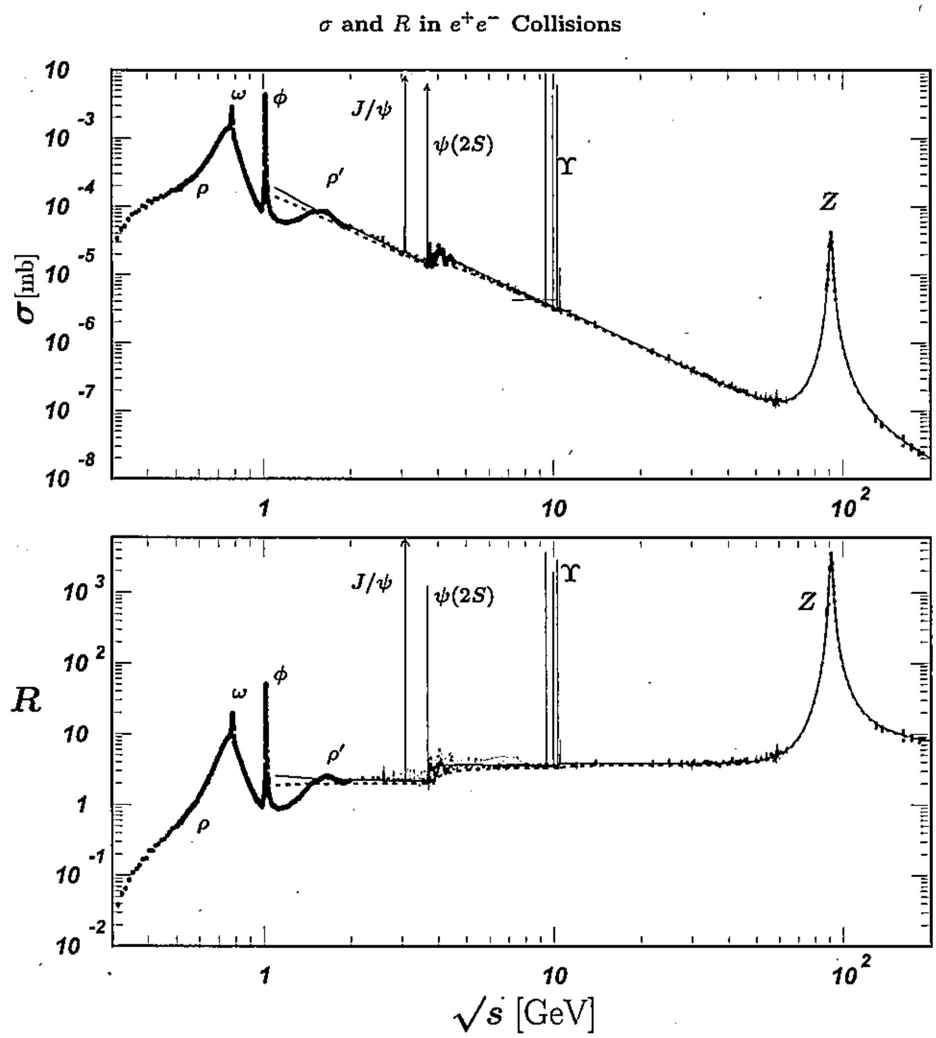


Figure 52.2: World data on the total cross section of $e^+e^- \to$ *hadrons* and the ratio $R(s) = \sigma(e^+e^- \to hadrons, s)/\sigma(e^+e^- \to \mu^+\mu^-, s)$. $\sigma(e^+e^- \to hadrons, s)$ is the experimental cross section corrected for initial state radiation and electron-positron vertex loops, $\sigma(e^+e^- \to \mu^+\mu^-, s) = 4\pi\alpha^2(s)/3s$. Data errors are total below 2 GeV and statistical above 2 GeV. The curves are an educative guide: the broken one (green) is a naive quark-parton model prediction, and the solid one (red) is 3-loop pQCD prediction (see "Quantum Chromodynamics" section of this *Review*, Eq. (9.7) or, for more details [99], Breit-Wigner parameterizations of J/ψ, $\psi(2S)$, and $\Upsilon(nS), n = 1, 2, 3, 4$ are also shown. The full list of references to the original data and the details of the R ratio extraction from them can be found in [100]. Corresponding computer-readable data files are available at http://pdg.lbl.gov/current/xsect/. (Courtesy of the COMPAS (Protvino) and HEPDATA (Durham) Groups, August 2019, Corrections by P. Janot (CERN) and M. Schmitt (Northwestern U.))

§4.2.2 Polarized case

In the high-energy limit of the $e^+e^- \to \mu^+\mu^-$ scattering, $m_e, m_\mu \ll \sqrt{s}$,

both $e^\pm$ and $\mu^\pm$ can be regarded as massless particles.

Dirac spinor for $p^\mu = (E, \vec{p}) \simeq (E, 0, 0, E)$

$$u_s(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar{\sigma}}\,\xi \end{pmatrix} \simeq \begin{pmatrix} \begin{pmatrix} 0 & \\ & \sqrt{2E} \end{pmatrix}\xi \\ \begin{pmatrix} \sqrt{2E} & \\ & 0 \end{pmatrix}\xi \end{pmatrix}$$

simplifies.

$$v_s(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ -\sqrt{p\cdot\bar{\sigma}}\,\xi \end{pmatrix} \simeq \begin{pmatrix} \begin{pmatrix} 0 & \\ & \sqrt{2E} \end{pmatrix}\xi \\ \begin{pmatrix} -\sqrt{2E} & \\ & 0 \end{pmatrix}\xi \end{pmatrix}$$

$\left(\gamma^0 = \begin{pmatrix} & \mathbb{1} \\ \mathbb{1} & \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \tau^i \\ -\tau^i & 0 \end{pmatrix} \text{ basis} \right)$

Then

$$\bar{v}_{s'}(\vec{p})\gamma^\nu u_r(\vec{p}) = \sqrt{2E_{\vec{p}}}\left(0,\ \xi_{s'}^\dagger \begin{pmatrix} 0 & \\ & -1 \end{pmatrix} \vec{\tau} \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} \xi_r + \xi_{s'}^\dagger \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} (-\vec{\tau}) \begin{pmatrix} 0 & \\ & 1 \end{pmatrix} \xi_r \right),$$

$$\bar{u}_r(\vec{p})\gamma^\mu v_s(-\vec{p}) = \sqrt{2E_{\vec{p}}}\left(0,\ \xi_r^\dagger \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} \vec{\tau} \begin{pmatrix} 0 & \\ & -1 \end{pmatrix} \xi_s + \xi_r^\dagger \begin{pmatrix} 0 & \\ & 1 \end{pmatrix} (-\vec{\tau}) \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} \xi_s \right),$$

when $\vec{p} \parallel \hat{e}_z$ (z-axis positive direction).

$$\Rightarrow (i\mathcal{M}) = \frac{i(Q_\mu Q_e e^2)}{s + i\varepsilon} s \times \mathrm{tr}_{2\times2}\left[\left(\vec{\tau} - \vec{n}_{in}(n_{in}\cdot\tau)\right)\left(\xi_{r'} \otimes \xi_{s'}^\dagger\right) \right] \times \mathrm{tr}_{2\times2}\left[\left(\vec{\tau} - \vec{n}_{out}(n_{out}\cdot\tau)\right)\left(\xi_s \otimes \xi_r^\dagger\right) \right].$$

$$\boxed{\begin{aligned}
&\boxed{e^-_{r'=\uparrow}\ e^+_{s'=\downarrow}} \Rightarrow \mathrm{tr}\left[\left(\vec{\tau} - \hat{e}_z(e_z\cdot\tau)\right) \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes (0, 1) \right) \right] = (1, i, 0) \Leftarrow \boxed{\mu^-_{r=\downarrow}\ \mu^+_{s=\uparrow}} \\
&\boxed{e^-_{r'=\downarrow}\ e^+_{s'=\uparrow}} \Rightarrow \mathrm{tr}\left[\left(\vec{\tau} - \hat{e}_z(e_z\cdot\tau)\right) \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes (1, 0) \right) \right] = (1, -i, 0) \Leftarrow \boxed{\mu^-_{r=\uparrow}\ \mu^+_{s=\downarrow}} \\
&\boxed{e^-_{r'=\uparrow}, e^+_{s'=\uparrow}}, \boxed{e^-_{r'=\downarrow}, e^+_{s'=\downarrow}} \Rightarrow (0, 0, 0) \Leftarrow \boxed{\mu^-_\uparrow \mu^+_\uparrow}\ \boxed{\mu^-_\downarrow \mu^+_\downarrow}
\end{aligned}}$$

When μ^- is moving out in the $(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) = \vec{n}_{out}$ direction,

multiply
$$\begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & & \sin\theta \\ & 1 & \\ -\sin\theta & & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\chi & -\sin\chi & 0 \\ \sin\chi & \cos\chi & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ on } \begin{pmatrix} 1 \\ \mp i \\ 0 \end{pmatrix}$$

to obtain $(\cos\theta\cos\varphi \pm i\sin\varphi,\ \cos\theta\sin\varphi \mp i\cos\varphi,\ -\sin\theta) \times e^{\pm i\chi}$.

So, $\mathrm{tr}_{2\times2}[\quad] \times \mathrm{tr}_{2\times2}[\quad] = \begin{cases} e^{i\varphi}e^{i\chi}\ (\cos\theta + 1) \\ e^{i\varphi}e^{-i\chi}\ (\cos\theta - 1) \\ e^{-i\varphi}e^{i\chi}\ (\cos\theta - 1) \\ e^{-i\varphi}e^{-i\chi}\ (\cos\theta + 1) \end{cases} = \begin{cases} e^{i(\varphi+\chi)}\, 2\cos^2(\theta/2) \\ e^{i(\varphi-\chi)}(-2)\sin^2(\theta/2) \\ e^{i(-\varphi+\chi)}(-2)\sin^2(\theta/2) \\ e^{-i(\varphi+\chi)}\, 2\cos^2(\theta/2) \end{cases}$

$\uparrow\downarrow \Rightarrow \uparrow\downarrow$	
$\uparrow\downarrow \Rightarrow \downarrow\uparrow$	
$\downarrow\uparrow \Rightarrow \uparrow\downarrow$	
$\downarrow\uparrow \Rightarrow \downarrow\uparrow$	
e^-e^+ ($\vec{n}_{in}$)	$\mu^-\mu^+$ ($\vec{n}_{out}$)

Therefore, the amplitudes for spin-polarized scatterings become

$$\left.\begin{array}{l} |\mathcal{M}(e^-_\uparrow e^+_\downarrow \to \mu^-_\uparrow \mu^+_\downarrow)|^2 \\ |\mathcal{M}(e^-_\uparrow e^+_\downarrow \to \mu^-_\downarrow \mu^+_\uparrow)|^2 \\ |\mathcal{M}(e^-_\downarrow e^+_\uparrow \to \mu^-_\uparrow \mu^+_\downarrow)|^2 \\ |\mathcal{M}(e^-_\downarrow e^+_\uparrow \to \mu^-_\downarrow \mu^+_\uparrow)|^2 \end{array}\right\} = \left\{\begin{array}{c} 4\cos^4(\theta/2) \\ 4\sin^4(\theta/2) \\ 4\sin^4(\theta/2) \\ 4\cos^4(\theta/2) \end{array}\right\} \times \left(e^2 Q(\mu) Q(e)\right)^2.$$

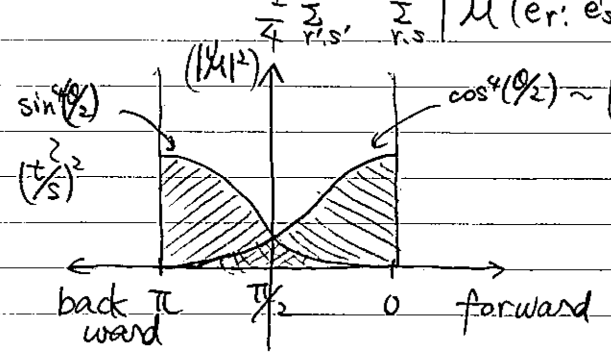
By summing up the spin in the final states,

$$\left.\begin{array}{l} \sum_{r,s} |\mathcal{M}(e^-_\uparrow e^+_\downarrow \to \mu^-_r, \mu^+_s)|^2 \\ \sum_{r,s} |\mathcal{M}(e^-_\uparrow e^+_\uparrow \to \mu^-_r, \mu^+_s)|^2 \\ \sum_{r,s} |\mathcal{M}(e^-_\downarrow e^+_\uparrow \to \mu^-_r, \mu^+_s)|^2 \\ \sum_{r,s} |\mathcal{M}(e^-_\downarrow e^+_\downarrow \to \mu^-_r, \mu^+_s)|^2 \end{array}\right\} = \left\{\begin{array}{c} 2(1+\cos^2\theta) \\ 0 \\ 2(1+\cos^2\theta) \\ 0 \end{array}\right\} \times \left(e^2 Q(\mu) Q(e)\right)^2.$$

By taking average over the spin in the initial states,

$$\frac{1}{4}\sum_{r',s'}\sum_{r,s} |\mathcal{M}(e^-_{r'} e^+_{s'} \to \mu^-_r, \mu^+_s)|^2 = (1+\cos^2\theta) \times \left(e^2 Q(\mu) Q(e)\right)^2$$

is reproduced.



$$e^-_\uparrow e^+_\downarrow \Rightarrow (\gamma^* \text{ with } j_z = +1) \Leftarrow \left\{\begin{array}{l} \mu^-_\uparrow \mu^+_\downarrow : j_{\hat{p}_3} = +1 \\ \mu^-_\downarrow \mu^+_\uparrow : j_{\hat{p}_3} = -1 \end{array}\right\}$$

$$e^-_\downarrow e^+_\uparrow \Rightarrow (\gamma^* \text{ with } j_z = -1)$$

sum & average

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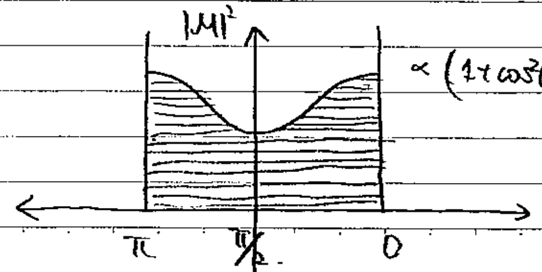

cf. see homework D-2 and/or E-1.
for a much easier (and theoretically interesting) method to compute scattering amplitudes of polarized massless particles