No

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§4.2.2 Polarized case

In the high-energy limit of the $e^+e^- \to \mu^+\mu^-$ scattering, $m_e, m_\mu \ll \sqrt{s}$,
both $e^\pm$ and $\mu^\pm$ can be regarded as massless particles.

Dirac spinor for $p^\mu = (E, \vec{p}) \simeq (E, 0, 0, E)$

$$u_s(\vec{p}) \simeq \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar\sigma}\,\xi \end{pmatrix} \simeq \begin{pmatrix} \begin{pmatrix} 0 & \\ & \sqrt{2E} \end{pmatrix}\xi \\ \begin{pmatrix} \sqrt{2E} & \\ & 0 \end{pmatrix}\xi \end{pmatrix}$$

simplifies.

$$v_s(\vec{p}) \simeq \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ -\sqrt{p\cdot\bar\sigma}\,\xi \end{pmatrix} \simeq \begin{pmatrix} \begin{pmatrix} 0 & \\ & \sqrt{2E} \end{pmatrix}\xi \\ \begin{pmatrix} -\sqrt{2E} & \\ & 0 \end{pmatrix}\xi \end{pmatrix}$$

$\left(\gamma^0 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \tau^i \\ -\tau^i & 0 \end{pmatrix} \text{ basis} \right)$

Then

$$\bar{v}_{s'}(\vec{p})\gamma^\nu u_{r'}(\vec{p}) = \sqrt{2E_{\vec p}}\left(0,\ \xi_{s'}^\dagger \begin{pmatrix} 0 & \\ & -1 \end{pmatrix} \vec{\tau} \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} \xi_{r'} + \xi_{s'}^\dagger \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} (-\vec{\tau}) \begin{pmatrix} 0 & \\ & 1 \end{pmatrix} \xi_{r'} \right),$$

$$\bar{u}_r(\vec{p})\gamma^\mu v_s(-\vec{p}) = \sqrt{2E_{\vec p}}\left(0,\ \xi_r^\dagger \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} \vec{\tau} \begin{pmatrix} 0 & \\ & -1 \end{pmatrix} \xi_s + \xi_r^\dagger \begin{pmatrix} 0 & \\ & 1 \end{pmatrix} (-\vec{\tau}) \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} \xi_s \right);$$

when $\vec{p} \parallel \hat{e}_z$ (z-axis positive direction).

$$\Longrightarrow (i\mathcal{M}) = \frac{i(Q_\mu Q_e e^2)}{s + i\varepsilon}\, s \times \mathrm{tr}_{2\times 2}\left[\left(\vec{\tau} - \vec{n}_{in}(n_{in}\cdot\tau)\right)\left(\xi_{r'} \otimes \xi_{s'}^\dagger\right) \right] \times \mathrm{tr}_{2\times 2}\left[\left(\vec{\tau} - \vec{n}_{out}(n_{out}\cdot\tau)\right)\left(\xi_s \otimes \xi_r^\dagger\right) \right].$$

$$\boxed{\begin{aligned} &\boxed{e^-_{r'=\uparrow}\ e^+_{s'=\downarrow}} \Rightarrow \mathrm{tr}\left[\left(\vec{\tau} - \hat{e}_z(e_z\cdot\tau)\right) \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes (0, 1)\right) \right] = (1, i, 0) \Leftarrow \boxed{\mu^-_{r=\downarrow}\ \mu^+_{s=\uparrow}} \\ &\boxed{e^-_{r'=\downarrow}\ e^+_{s'=\uparrow}} \Rightarrow \mathrm{tr}\left[\left(\vec{\tau} - \hat{e}_z(e_z\cdot\tau)\right) \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes (1, 0)\right) \right] = (1, -i, 0) \Leftarrow \boxed{\mu^-_{r=\uparrow}\ \mu^+_{s=\downarrow}} \\ &\boxed{e^-_{r'=\uparrow}, e^+_{s'=\uparrow}},\ \boxed{e^-_{r'=\downarrow}, e^+_{s'=\downarrow}} \Rightarrow (0, 0, 0) \Leftarrow \boxed{\mu^-_\uparrow \mu^+_\uparrow}\ \boxed{\mu^-_\downarrow \mu^+_\downarrow} \end{aligned}}$$

When μ^- is moving out in the $(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) = \vec{n}_{out}$ direction

multiply
$$\begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & & \sin\theta \\ & 1 & \\ -\sin\theta & & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\chi & -\sin\chi & 0 \\ \sin\chi & \cos\chi & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ on } \begin{pmatrix} 1 \\ \mp i \\ 0 \end{pmatrix}$$

to obtain $(\cos\theta\cos\varphi \pm i\sin\varphi,\ \cos\theta\sin\varphi \mp i\cos\varphi,\ -\sin\theta) \times e^{\pm i\chi}$.

$$\text{So, } \mathrm{tr}_{2\times 2}[\ ] \times \mathrm{tr}_{2\times 2}[\ ] = \begin{cases} e^{i\varphi}e^{i\chi}\,(\cos\theta + 1) \\ e^{i\varphi}e^{-i\chi}\,(\cos\theta - 1) \\ e^{-i\varphi}e^{i\chi}\,(\cos\theta - 1) \\ e^{i\varphi}e^{-i\chi}\,(\cos\theta + 1) \end{cases} = \begin{cases} e^{i(\varphi+\chi)}\,2\cos^2(\theta/2) \\ e^{i(\varphi-\chi)}(-2)\sin^2(\theta/2) \\ e^{i(-\varphi+\chi)}(-2)\sin^2(\theta/2) \\ e^{-i(\varphi+\chi)}\,2\cos^2(\theta/2) \end{cases} \left| \begin{array}{l} \uparrow\downarrow \Rightarrow \uparrow\downarrow \\ \uparrow\downarrow \Rightarrow \downarrow\uparrow \\ \downarrow\uparrow \Rightarrow \uparrow\downarrow \\ \downarrow\uparrow \Rightarrow \downarrow\uparrow \\ \begin{matrix} e^-e^+ & \mu^-\mu^+ \\ (\vec{n}_{in}) & (\vec{n}_{out}) \end{matrix} \end{array} \right.$$

Therefore, the amplitudes for spin-polarized scatterings become

$$\left.\begin{matrix} |\mathcal{M}(e^-_\uparrow e^+_\downarrow \to \mu^-_\uparrow \mu^+_\downarrow)|^2 \\ |\mathcal{M}(e^-_\uparrow e^+_\downarrow \to \mu^-_\downarrow \mu^+_\uparrow)|^2 \\ |\mathcal{M}(e^-_\downarrow e^+_\uparrow \to \mu^-_\uparrow \mu^+_\downarrow)|^2 \\ |\mathcal{M}(e^-_\downarrow e^+_\uparrow \to \mu^-_\downarrow \mu^+_\uparrow)|^2 \end{matrix}\right\} = \left\{\begin{matrix} 4\cos^4(\theta/2) \\ 4\sin^4(\theta/2) \\ 4\sin^4(\theta/2) \\ 4\cos^4(\theta/2) \end{matrix}\right\} \times \left(e^2 Q(\mu) Q(e)\right)^2.$$

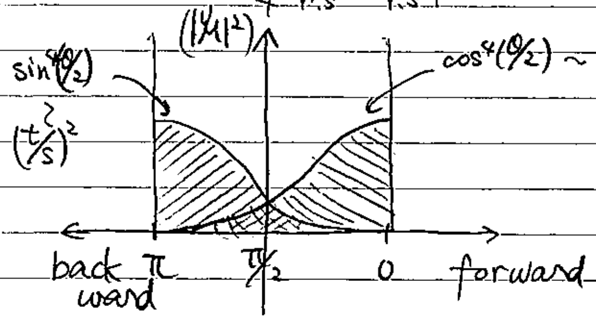
By summing up the spin in the final states,

$$\left.\begin{matrix} \sum_{r,s} |\mathcal{M}(e^-_\uparrow e^+_\downarrow \to \mu^-_r \mu^+_s)|^2 \\ \sum_{r,s} |\mathcal{M}(e^-_\uparrow e^+_\uparrow \to \mu^-_r \mu^+_s)|^2 \\ \sum_{r,s} |\mathcal{M}(e^-_\downarrow e^+_\uparrow \to \mu^-_r \mu^+_s)|^2 \\ \sum_{r,s} |\mathcal{M}(e^-_\downarrow e^+_\downarrow \to \mu^-_r \mu^+_s)|^2 \end{matrix}\right\} = \left\{\begin{matrix} 2(1+\cos^2\theta) \\ 0 \\ 2(1+\cos^2\theta) \\ 0 \end{matrix}\right\} \times \left(e^2 Q(\mu) Q(e)\right)^2.$$

By taking average over the spin in the initial states,

$$\frac{1}{4}\sum_{r',s'}\sum_{r,s} |\mathcal{M}(e^-_{r'} e^+_{s'} \to \mu^-_r \mu^+_s)|^2 = (1+\cos^2\theta)\times\left(e^2 Q(\mu) Q(e)\right)^2$$

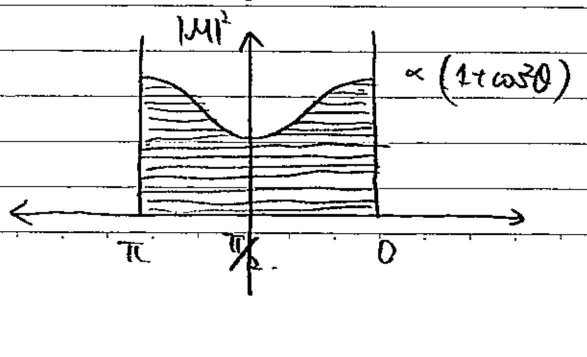
is reproduced.



$$e^-_\uparrow e^+_\downarrow \Rightarrow (\gamma^* \text{ with } j_z = +1) \Leftarrow \left\{\begin{matrix} \mu^-_\uparrow \mu^+_\downarrow : j_{\hat{p}_3} = +1 \\ \mu^-_\downarrow \mu^+_\uparrow : j_{\hat{p}_3} = -1 \end{matrix}\right\}$$

$$e^-_\downarrow e^+_\uparrow \Rightarrow (\gamma^* \text{ with } j_z = -1)$$

↓ sum & average



cf. see homework D-1 and/or E-1 for a much easier (and theoretically interesting) method to compute scattering amplitudes of polarized massless particles

§ 4.3 Crossing symmetry

Consider a scalar QED., based on the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + (D_\mu \Phi_i)^* (D^\mu \Phi_i) - (M_i)^2 |\Phi_i|^2 ;$$

where Φ_i is a complex scalar field.

$$D_\mu = (\partial_\mu + i e Q_i A_\mu). \qquad (e>0)$$

Then

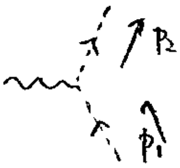
$$\mathcal{L}_{int} = \sum_i e A_\mu \cdot Q_i \left[(+i\partial^\mu \Phi_i^*) \Phi_i - \Phi_i^* (i\partial^\mu \Phi_i) \right] + e^2 A_\mu A^\mu \sum_i \left[Q_i^2 \Phi_i^* \Phi_i \right].$$

Using the fact that

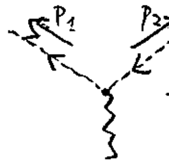
$$\left.\begin{aligned} \Phi_i(x) &= \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec p}}} \left(a_{\vec p}\, \underline{\underline{e^{-ip\cdot x}}} + b^\dagger_{\vec p}\, \underline{\underline{e^{ip\cdot x}}} \right) \\ \Phi_i^\dagger(x) &= \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec p}}} \left(a^\dagger_{\vec p}\, \underline{\underline{e^{ip\cdot x}}} + b_{\vec p}\, \underline{\underline{e^{-ip\cdot x}}} \right) \end{aligned}\right\} \longleftarrow$$

p^μ: on the mass-shell ($p^2 = M_i^2$) with $p^{\mu=0} > 0$.

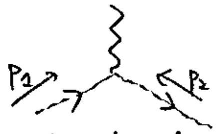
one can derive that



$$\Rightarrow -ieQ_i(p_1+p_2)^\mu$$

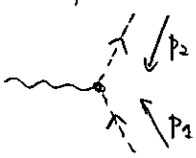


$$\Rightarrow -ieQ_i(p_1-p_2)^\mu$$



$$\Rightarrow -ieQ_i(p_1-p_2)^\mu$$

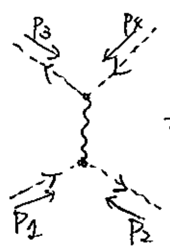
All of those vertex factors are in the form of



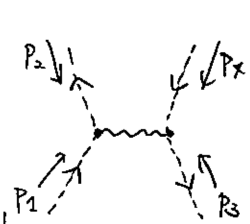
$$\Rightarrow -ieQ_i(p_1-p_2)^\mu$$

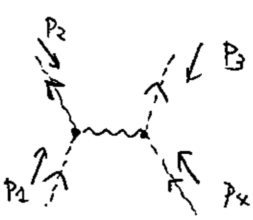
when all the momenta (not necessarily $p^0>0$) are defined to be flowing <u>in</u> to the vertex.

This means that



$$\Rightarrow i\mathcal{M} = \frac{i(eQ)^2\,(p_1-p_2)\cdot(p_4-p_3)}{(p_1+p_2)^2+i\varepsilon} \Leftarrow$$





$\begin{cases} \cdot\ (p_3)^0, (p_4)^0 < 0 \\ \cdot\ (p_1+p_2)^2 > 0 \end{cases}$

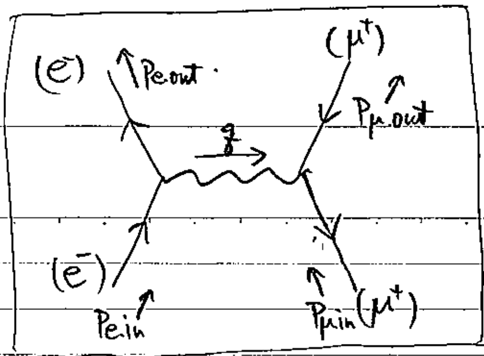
this common amplitude is used via analytic continuation at different regions of the kinematical variables. = <u>crossing symmetry</u>

$\begin{cases} \cdot\ (p_2)^0, (p_4)^0 < 0 \\ \cdot\ (p_1+p_2)^2 < 0 \end{cases}$, $\begin{cases} \cdot\ (p_2)^0, (p_3)^0 < 0 \\ \cdot\ (p_1+p_2)^2 < 0 \end{cases}$

§3.4

$(e^- \mu^+ \longrightarrow e^- \mu^+)$

$p_{e.in}$ $p_{\mu.in}$ $\quad$ $p_{e.out}$ $p_{\mu.out}$



The time-ordered correlation fcn at $\mathcal{O}(e^2)$ [fully connected cmpn't]

is given by

$$\int \frac{i(\not{p}_{e.out}+m)}{(p_{e.out}^2-m^2+i\varepsilon)}(-iQ_e e\gamma^\mu)\frac{i(\not{p}_{e.in}+m)}{(p_{e.in}^2-m^2+i\varepsilon)} \quad \frac{-i\eta_{\mu\nu}}{(q^2+i\varepsilon)}$$

$$\frac{i(-\not{p}_{\mu.out}+M)}{(p_{\mu.out}^2-M^2+i\varepsilon)}(+iQ_\mu e\gamma^\nu)\frac{i(-\not{p}_{\mu.in}+M)}{(p_{\mu.in}^2-M^2+i\varepsilon)}$$

$$(2\pi)^4\delta^4(p_{e.out}+q-p_{e.in})\times(2\pi)^4\delta^4(p_{\mu.out}-q-p_{\mu.in})\,\frac{d^4q}{(2\pi)^4}.$$

By extracting the residue, we obtain the scattering amplitude

$$\boxed{i\mathcal{M} = -i(Q_eQ_\mu e^2)\left[\bar{u}(\vec{p}_{e.out})\gamma^\mu u(\vec{p}_{e.in})\right]\left[\bar{v}(\vec{p}_{\mu.in})\gamma_\mu v(\vec{p}_{\mu.out})\right]\frac{1}{(p_{e.out}-p_{e.in})^2+i\varepsilon}}$$

After summing up the final state spins and averaging the initial state spins, we have

$$\frac{1}{4}\sum_{r's'}\sum_{r.s}|\mathcal{M}|^2 = \frac{(Q_eQ_\mu e^2)^2}{4t^2}\,\mathrm{Tr}_{4\times4}\left[\gamma^\mu(\not{p}_{e.in}+m)\gamma^\kappa(\not{p}_{e.out}+m)\right]$$

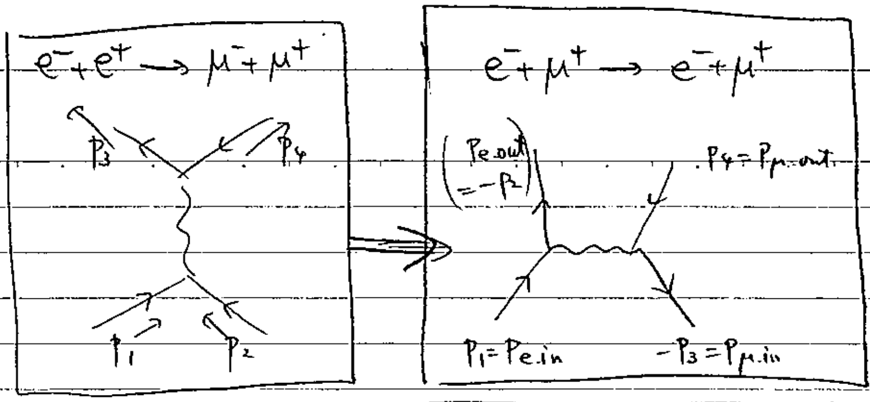
$$\mathrm{Tr}_{4\times4}\left[\gamma_\mu(\not{p}_{\mu.out}-M)\gamma_\kappa(\not{p}_{\mu.in}-M)\right]$$

$$= \frac{(Q_\mu Q_e e^2)^2}{t^2}4\left\{m^2\eta^{\mu\kappa}-(p_{e.in}\cdot p_{e.out})\eta^{\mu\kappa}+(p_{e.in}^\mu p_{e.out}^\kappa + p_{e.out}^\mu p_{e.in}^\kappa)\right\}$$

$$\left\{M^2\eta_{\mu\kappa}-(p_{\mu.in}\cdot p_{\mu.out})\eta_{\mu\kappa}+(p_{\mu.out,\mu}p_{\mu.in,\kappa}+p_{\mu.out,\kappa}p_{\mu.in,\mu})\right\}$$

$$= \frac{(Q_\mu Q_e e^2)^2}{t^2}4\left[0\cdot t^2 + 2\left\{\left(\frac{-u+M^2+m^2}{2}\right)^2+\left(\frac{s-M^2-m^2}{2}\right)^2\right\} + t\left(-\frac{t}{2}+M^2\right)+t\left(-\frac{t}{2}+m^2\right)\right]$$

$$= \frac{(Q_\mu Q_e e^2)^2}{t^2}4\left[\frac{u^2+s^2}{2}+(t-u-s)(m^2+M^2)+(m^2+M^2)^2\right]$$



$$\boxed{\begin{aligned} s &= (p_1+p_2)^2 \Rightarrow (p_{e.in} - p_{e.out})^2 = t \\ t &= (p_3-p_1)^2 \Rightarrow (p_{e.in} + p_{\mu.in})^2 = s \\ u &= (p_4-p_1)^2 \Rightarrow (p_{\mu.out} - p_{e.in})^2 = u . \end{aligned}}$$

$$\frac{1}{4}\sum_{r',s',r,s} |\mathcal{M}|^2 \Big/ \left[(Q_\mu Q_e e^2)^2 4\right]$$

$$= \frac{1}{s^2}\left[\frac{u^2+t^2}{2} + (s-u-t)(m^2+M^2) + (m^2+M^2)\right] \Longleftrightarrow \frac{1}{t^2}\left[\frac{u^2+s^2}{2} + (t-u-s)(m^2+M^2) + (m^2+M\right.$$

for $e^- + e^+ \to \mu^- + \mu^+$ $\qquad\qquad$ $e^- + \mu^+ \to e^- + \mu^+$.

<u>crossing symmetry</u>

Consider the non-relativistic limit of the current in a scattering.

$[\bar{u}_r(\vec{p}\,')\gamma^\mu u_s(\vec{p})]$ where the 4-momenta $p^\mu=(E_{\vec p},\vec p)$ and $p'^\mu=(E_{\vec p'},\vec p\,')$ are both non-relativistic.

Now, in the basis where $\gamma^\mu=\begin{pmatrix} & \sigma^\mu \\ \bar\sigma^\mu & \end{pmatrix}$ $\quad\sigma^\mu=(\mathbb{1},\vec\tau)$, $\bar\sigma^\mu=(\mathbb{1},-\vec\tau)$ as in P.S.

$$u_s(\vec p)=\begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi_s\\ \sqrt{p\cdot\bar\sigma}\,\xi_s\end{pmatrix}\cong\begin{pmatrix}\sqrt{\left(M+\frac{\vec p^{\,2}}{2M}-\vec p\cdot\vec\tau\right)}\,\xi_s\\ \sqrt{\left(M+\frac{\vec p^{\,2}}{2M}+\vec p\cdot\vec\tau\right)}\,\xi_s\end{pmatrix}$$

$$\cong\sqrt{M}\begin{pmatrix}\left(1-\frac{\vec p\cdot\vec\tau}{2M}+\frac{(\vec p)^2}{8M^2}\right)\xi_s\\ \left(1+\frac{\vec p\cdot\vec\tau}{2M}+\frac{(\vec p)^2}{8M^2}\right)\xi_s\end{pmatrix}.$$

So,

- $$[\bar{u}_r(\vec p\,')\gamma^0 u_s(\vec p)]\cong M\times\left(\begin{array}{l}\xi_r^\dagger\left(1+\frac{\vec p\,'\cdot\vec\tau}{2M}+\frac{(\vec p\,')^2}{8M^2}\right)\cdot\mathbb{1}_{2\times2}\cdot\left(1+\frac{\vec p\cdot\vec\tau}{2M}+\frac{(\vec p)^2}{8M^2}\right)\xi_s\\ +\,\xi_r^\dagger\left(1-\frac{\vec p\,'\cdot\vec\tau}{2M}+\frac{(\vec p\,')^2}{8M^2}\right)\cdot\mathbb{1}_{2\times2}\cdot\left(1-\frac{\vec p\cdot\vec\tau}{2M}+\frac{(\vec p)^2}{8M^2}\right)\xi_s\end{array}\right)$$

$$\cong 2M\left\{\xi_r^\dagger\xi_s\left(1+\frac{(\vec p\,')^2+(\vec p)^2}{8M^2}\right)+\xi_r^\dagger\frac{(\vec p\,'\cdot\vec\tau)(\vec p\cdot\vec\tau)}{4M^2}\xi_s\right\}+\mathcal{O}(|\vec p|^3)$$

$$=2M\left\{\xi_r^\dagger\xi_s\left(1+\frac{(\vec p_{av})^2}{2M^2}\right)+\frac{i(\vec p\,'-\vec p)^i(\vec p_{av})^j\epsilon^{ijk}}{2M^2}\left(\xi_r^\dagger\frac{\tau^k}{2}\xi_s\right)\right\}\qquad \vec p_{av}:=\frac{\vec p\,'+\vec p}{2}$$

- $$[\bar{u}_r(\vec p\,')\gamma^i u_s(\vec p)]\cong M\left\{\begin{array}{l}\xi_r^\dagger\left(1+\frac{\vec p\,'\cdot\vec\tau}{2M}+\cdots\right)\cdot\tau^i\cdot\left(1+\frac{\vec p\cdot\vec\tau}{2M}+\cdots\right)\xi_s\\ -\,\xi_r^\dagger\left(1-\frac{\vec p\,'\cdot\vec\tau}{2M}+\cdots\right)\cdot\tau^i\cdot\left(1-\frac{\vec p\cdot\vec\tau}{2M}+\cdots\right)\xi_s\end{array}\right\}$$

$$\cong 2M\,\xi_r^\dagger\frac{(\vec p\,'\cdot\vec\tau)\tau^i+\tau^i(\vec p\cdot\vec\tau)}{2M}\xi_s$$

$$=2M\left\{(\xi_r^\dagger\xi_s)\cdot\frac{p_{av}^i}{M}-i\epsilon^{ijk}\frac{\overset{(\vec p\,'-\vec p)}{(\Delta p)^j}}{M}\left[\xi_r^\dagger\left(\frac{\tau^k}{2}\right)\xi_s\right]\right\}$$

$\Rightarrow$ In a t-channel scattering, a non-rela particle rarely changes its spin.

$\Rightarrow$ $[\bar u\gamma^\mu u]$ in the quantum scattering amplitude is approximately the classical $\boxed{2M\times u^\mu=2\cdot p^\mu_{av.}}$ apart from spin-dep. corrections suppressed by $\left(\frac{\Delta\vec p}{M}\right)$.

No.
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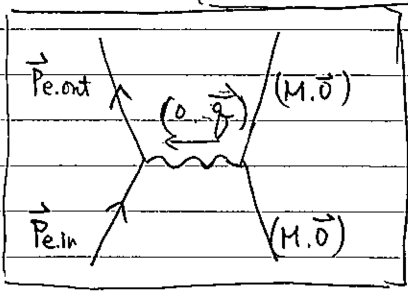
☆ Let us take the target (μ^+ now) mass M to be much larger than the incoming electron energy. (in the CM frame).

Then
$$\left\{\begin{array}{ll}\left[\bar{u}_r(\vec{p}_{\mu.\text{out}})\gamma^\mu u_{r'}(\vec{p}_{\mu.\text{in}})\right] & \text{target } \mu^- \\ \left[\bar{v}_{r'}(\vec{p}_{\mu.\text{in}})\gamma^\mu v_r(\vec{p}_{\mu.\text{out}})\right] & \text{taarget } \mu^+\end{array}\right\}$$
is approximated by $2M_\mu \times (1.0.0.0)$,

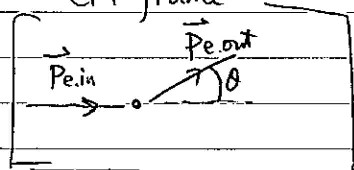
and the photon propagator in the Feynmangauge
$$\left(\frac{-i\eta_{\mu\nu}}{t} = \frac{-i\eta_{\mu\nu}}{-|\vec{q}|^2}\right)$$
coupled with the current $\propto Q_\mu \cdot e \cdot (1.\vec{0})^\nu$
gives rise to the Fourier transform of the Coulomb potential
(as we are familiar with in Quantum Mechanics).
$\hookrightarrow$ regardless of the spin of the target particle.



If we sum over the spin of the final state e^-, and average over the spin of the initial state e^-,

then
$$\frac{1}{2}\sum_{s.s'}\left|\mathcal{M}(e^-\mu^+\to e^-\mu^+)\right|^2 \simeq \frac{(Q_\mu Q_e e^2)^2}{(|\vec{q}|^2)^2}\frac{(2M_\mu)^2}{2}\,\mathrm{Tr}_{4\times4}\left[\gamma^0(\not{p}_{e.\text{in}}+m_e)\gamma^0(\not{p}_{e.\text{out}}+m)\right]$$
$$\simeq \frac{e^4}{|\vec{q}|^4}(2M_\mu)^2\,2\left[p^0_{e.\text{in}}p^0_{e.\text{out}} + \vec{p}_{e.\text{in}}\cdot\vec{p}_{e.\text{out}} + m_e^2\right]$$
$$= \frac{e^4\,(2M_\mu)^2}{|\vec{p}_e|^4\left(2\sin\left(\frac{\theta}{2}\right)\right)^4}\cdot 4\left[E_e^{\,2} - |p_e|^2\sin^2\left(\frac{\theta}{2}\right)\right]$$

CM frame



$E_{e.\text{in}} \simeq E_{e.\text{out}}$
$|\vec{p}_{e.\text{in}}| \simeq |\vec{p}_{e.\text{out}}|$
(recoil of the target negligible)

The final state phase space is
$$\frac{1}{2M\cdot 2E_e\cdot\left(p_e/E_e - 0\right)}\int\frac{d^3\vec{p}_{e.\text{out}}}{(2\pi)^3}\frac{1}{2E_e}\frac{1}{2M_\mu}(2\pi)\,\delta(E_{e^+_{\text{out}}} - E_{e.\text{in}})$$
$$\simeq \frac{d^2\Omega}{(4\pi)^2}\frac{1}{4M_\mu^2}.$$

So.
$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha_e^2}{4|p_e|^4\sin^4(\theta/2)}\left[E_e^{\,2} - p_e^{\,2}\sin^2(\theta/2)\right]} \xrightarrow{e\ \text{non rela}} \boxed{\frac{\alpha_e^2\ m_e^2}{4|p_e|^4\sin^4(\theta/2)}}$$
(Rutherford scattering)