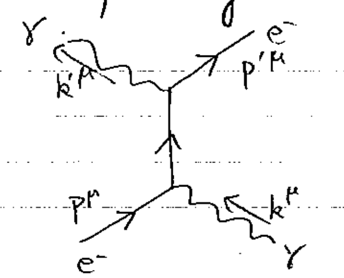
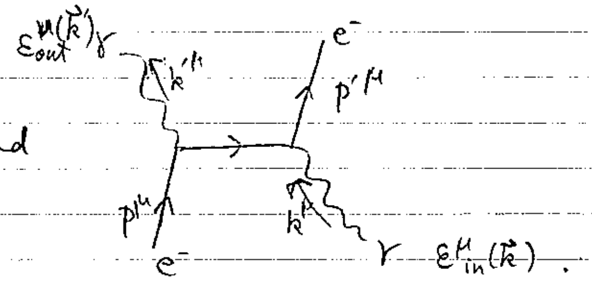
§ 3.5 Compton / Thomson scattering

$$e^-_{r'}(p^\mu) + \gamma_{s'}(k^\mu) \longrightarrow e^-_{r}(p'^\mu) + \gamma_{s}(k'^\mu)$$ 2-to-2 elastic scattering

There are two contributions to the scattering amplitude corresponding to



and



$$i\mathcal{M} = (ieQ_{(e)})^2 \left\{ \frac{\bar{u}_r(\vec{p}')\,\gamma^\nu\, i[(\not{p}+\not{k})+m]\,\gamma^\mu\, u_{r'}(\vec{p})}{((p+k)^2-m^2+i\epsilon)} + \frac{\bar{u}_r(\vec{p}')\,\gamma^\mu\, i[(\not{p}-\not{k}')+m]\,\gamma^\nu\, u_{r'}(\vec{p})}{((p-k')^2-m^2+i\epsilon)} \right\} \varepsilon^*_{\nu\,\mathrm{out},s}(\vec{k}')\;\varepsilon_{\mu\,\mathrm{in},s'}(\vec{k}).$$

The relative sign is "+" because the contraction pattern for the 2nd line

$(\bar\psi\gamma_\nu\psi)(\bar\psi\gamma_\mu\psi)$ can be brought into $(\bar\psi\gamma_\nu\psi)(\bar\psi\gamma_\mu\psi)$

by an even permutation of the fermion fields.

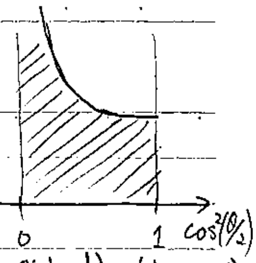
Now, an almost straightforward (but lengthy) calculation brings us to

$$\frac{1}{2}\sum_{r'=1}^{2}\frac{1}{2}\sum_{s'=1}^{2}\sum_{r,s=1}^{2}|\mathcal{M}|^2 = 2e^4\left[\frac{p\cdot k'}{p\cdot k}+\frac{p\cdot k}{p\cdot k'}+2m^2\left(\frac{1}{p\cdot k}-\frac{1}{p\cdot k'}\right)+m^4\left(\frac{1}{p\cdot k}-\frac{1}{(p\cdot k')}\right)^2\right]$$

(*) see Peskin Schroeder §5.5.

☆ High-energy limit ($\sqrt{s} \gg m_e$)

(θ: CM frame scattering angle)



$$(*) \cong 2e^4\left(\frac{-u}{s} + \frac{s}{-u}\right) = 2e^4 \frac{u^2+s^2}{(-su)} = 2e^4 \frac{1+\cos^4(\theta/2)}{\cos^2(\theta/2)}$$

$$\frac{d\sigma}{d\cos\theta} \cong \underset{\Uparrow\ \text{phase space integral}}{\frac{1}{32\pi s}} \times (*) = \frac{\pi\alpha^2}{s}\left[\frac{1+\cos^4(\theta/2)}{\cos^2(\theta/2)}\right] = \frac{\pi\alpha^2}{2s}\left[\frac{4+(\cos\theta+1)^2}{(\cos\theta+1)}\right].$$

An apparently divergent integral $\int_{-1} d(\cos\theta) \frac{1}{(\cos\theta+1)}$ is finite in fact, because the denominator of the $\frac{s}{-u}$ term in (*) is actually

$(2p\cdot k') = -u + m^2$; $(-u) \sim s\cdot\cos^2(\theta/2)$.

The pole $\frac{1}{(\cos\theta+1)} = \frac{2}{\cos^2(\theta/2)}$ in the integrand should be modified at $\cos^2(\theta/2) \lesssim m^2/s$. $\underbrace{\sigma_{tot} \sim \frac{2\pi\alpha^2}{s}\cdot \ln(s/m^2)}$.

☆ Non-relativistic limit ($\sqrt{s} \ll m_e$)

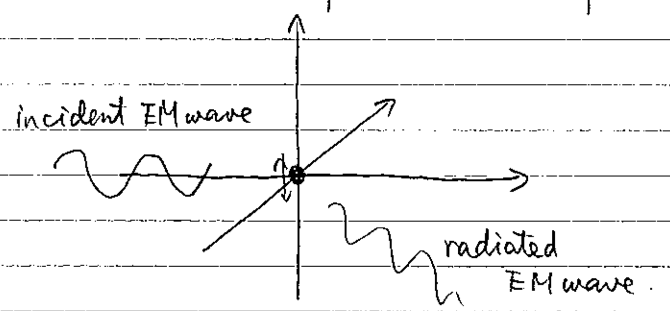
At the center of mass frame $\left\{\begin{matrix} p^\mu \simeq (m, -\vec{k}) & p'^\mu \simeq (m, -\vec{k}') \\ k^\mu = (k, \vec{k}) & k'^\mu = (k', \vec{k}') \end{matrix}\right\}$. $(k' = k = |\vec{k}|)$

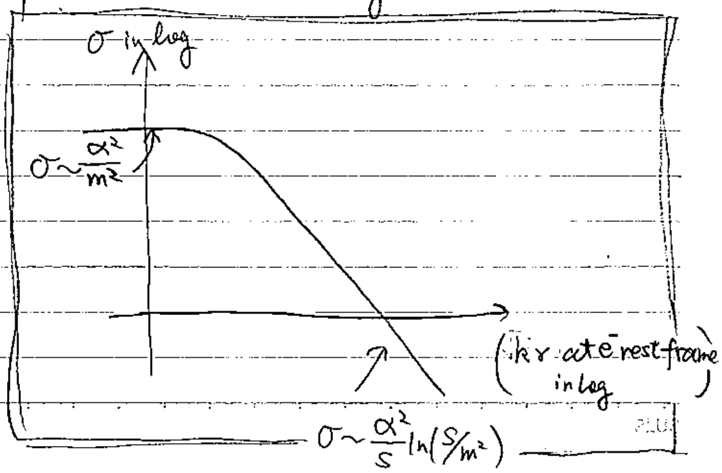
$(p\cdot k) = mk + k^2$ and $(p\cdot k') = mk + k^2\cos\theta$.

$$(*) \cong 2e^4\left[1 + 1 + 2(\cos\theta - 1) + (\cos\theta-1)^2\right] = 2e^4\left[1+\cos^2\theta\right].$$

$$d\sigma = \frac{1}{2k\cdot 2m\cdot 1}\,\frac{d\cos\theta}{(2\pi)}\,\frac{k^2}{2k\cdot 2m} \times (*) = \boxed{\frac{d(\cos\theta)}{32\pi\, m^2}} \times 2e^4\left[1+\cos^2\theta\right] = d\cos\theta\,\frac{\pi\alpha^2}{m_e^2}(1+\cos^2\theta).$$

This reproduces the formula of Thomson scattering.





Supplementary notes.

Gamma matrices γ^μ only need to satisfy the relation $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}_{4\times4}$

There are many expressions, which are connected to one another by similarity transformations.

Take

$$\gamma^0_{PS} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \gamma^i_{PS} = \begin{pmatrix} 0 & \tau^i \\ -\tau^i & 0 \end{pmatrix}$$ first, as in Peskin-Schroeder.

Let the Dirac spinor in this frame be Ψ_{PS}.

The non-rela expression of the Gamma matrices are

$$\gamma^0_{\text{non-R}} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \gamma^i_{\text{non-R}} = \begin{pmatrix} 0 & \tau^i \\ -\tau^i & 0 \end{pmatrix}$$, on the other hand.

One can verify that

$$\frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1} & \mathbb{1} \\ -\mathbb{1} & \mathbb{1} \end{pmatrix} \gamma^\mu_{PS} \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1} & -\mathbb{1} \\ \mathbb{1} & \mathbb{1} \end{pmatrix} = \gamma^\mu_{\text{non-R}} .$$

So,

$$\Psi_{PS} = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1} & -\mathbb{1} \\ \mathbb{1} & \mathbb{1} \end{pmatrix} \cdot \Psi_{\text{non-R}} .$$

$$\Psi_{\text{non-R}} = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1} & \mathbb{1} \\ -\mathbb{1} & \mathbb{1} \end{pmatrix} \Psi_{PS} \xrightarrow[\text{limit}]{\text{non-relati.}} \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1} & \mathbb{1} \\ -\mathbb{1} & \mathbb{1} \end{pmatrix} \begin{pmatrix} \sqrt{m}\left(1 - \frac{\vec{p}\cdot\vec{\tau}}{2m} + \frac{\vec{p}^2}{8m^2} + \cdots\right)\xi \\ \sqrt{m}\left(1 + \frac{\vec{p}\cdot\vec{\tau}}{2m} + \frac{\vec{p}^2}{8m^2} + \cdots\right)\xi \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{2m}\left(\mathbb{1} + \frac{|\vec{p}|^2}{8m^2} + \cdots\right)\xi \\ \sqrt{2m}\left(\frac{\vec{p}\cdot\vec{\tau}}{2m} + \cdots\right)\xi \end{pmatrix} .$$

Thomson scattering in QED without spin sum/average

It is convenient to use the non-relativistic frame for γ^μ's & Ψ.

- $$\varepsilon^*_{i,s}\,\varepsilon_{j,s'}\,\bar{u}_r(\vec{p}\,')\,\gamma^i\,[(\cancel{p+k})+m]\,\gamma^j\,u_{r'}(\vec{p}) \simeq \xi_r^\dagger\cdot\left(\mathbb{1},\ -\frac{\vec{p}\,'\cdot\vec{\tau}}{2m}\right)\begin{bmatrix}0 & -\varepsilon^*_s\cdot\tau\\ \varepsilon^*_s\cdot\tau & 0\end{bmatrix}\begin{bmatrix}2m+k & -(\vec{p}+\vec{k})\cdot\vec{\tau}\\ (\vec{p}+\vec{k})\cdot\vec{\tau} & -k\end{bmatrix}\begin{bmatrix} & -\varepsilon_{s'}\cdot\tau\\ \varepsilon_{s'}\cdot\vec{\tau} & \end{bmatrix}\begin{pmatrix}\mathbb{1}\\ \frac{\vec{p}\cdot\vec{\tau}}{2m}\end{pmatrix}\cdot\xi_{r'}$$
$$\times(2m)$$
$$\cong (2m)\,k\,(\xi_r^\dagger\,\tau^i\,\tau^j\,\xi_{r'})\,\varepsilon^{i*}_s\,\varepsilon^j_{s'}$$

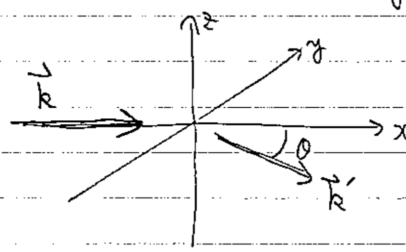
- $$\varepsilon^*_{i,s}\,\varepsilon_{j,s'}\,\bar{u}_r(\vec{p}\,')\,\gamma^j\,[(\cancel{p-k'})+m]\,\gamma^i\,u_{r'}(\vec{p}) \simeq \xi_r^\dagger\cdot\left(1,\ -\frac{\vec{p}\,'\cdot\vec{\tau}}{2m}\right)\begin{bmatrix}0 & -\varepsilon_{s'}\cdot\tau\\ \varepsilon_{s'}\cdot\tau & 0\end{bmatrix}\begin{bmatrix}2m-k & -(\vec{p}-\vec{k}')\vec{\tau}\\ (\vec{p}-\vec{k}')\cdot\vec{\tau} & +k\end{bmatrix}\begin{bmatrix}0 & -\varepsilon^*_s\cdot\tau\\ \varepsilon^*_s\cdot\tau & 0\end{bmatrix}\begin{pmatrix}1\\ \frac{\vec{p}\cdot\vec{\tau}}{2m}\end{pmatrix}\xi_{r'}$$
$$\times(2m)$$
$$\cong -(2m)\,k\,(\xi_r^\dagger\,\tau^j\,\tau^i\,\xi_{r'})\,\varepsilon^{i*}_s\,\varepsilon^j_{s'}$$

- denominators : $(p+k)^2-m^2 = 2p\cdot k \simeq 2mk$
  $(p-k')^2-m^2 = -2p\cdot k' \simeq -2mk$

- by summing up…

$$i\mathcal{M} = -ie^2\left\{\frac{2mk\,(\xi_r^\dagger\tau^i\tau^j\xi_{r'})}{2mk}+\frac{-2mk\,(\xi_r^\dagger\tau^j\tau^i\xi_{r'})}{(-2mk)}\right\}\varepsilon^{i*}_s\,\varepsilon^j_{s'} = -i2e^2\,(\xi_r^\dagger\xi_{r'})(\vec{\varepsilon}_s\cdot\vec{\varepsilon}_{s'})$$

When the incident EM wave has a polarization $\vec{\varepsilon}_{s'}$ perpendicular to the scattering plane, $(\vec{\varepsilon}_{s'}=\hat{e}_z)$ we can take $\vec{\varepsilon}_s=\vec{\varepsilon}_{s'}$. $\sum_s|\vec{\varepsilon}_s\cdot\vec{\varepsilon}_{s'}|^2 = 1$.



If $\vec{\varepsilon}_{s'}$ is within the scattering plane $(\vec{\varepsilon}_{s'}=\hat{e}_x)$,

then $\sum_s|\vec{\varepsilon}_s\cdot\vec{\varepsilon}_{s'}|^2 = \cos^2\theta$.

The "1" and "$\cos^2\theta$" components in the Thomson scattering formula correspond to those polarizations.

$\sin^2\theta$

$1+\sin^2\left(\frac{\pi}{2}-\theta\right)$

§4 Bound States

§4.1 Bethe – Salpeter equation

Consider non-relativistic particles

$$\mathcal{L} = \psi_a^\dagger \left(i\partial_t + \frac{\partial_x^2}{2m_a} - eQ_a\varphi - m_a \right)\psi_a$$

(non-rela. limit of Dirac fermion or complex boson)

Think of $e^- p^+$, $e^- \mu^+$, $e^- e^+$ bound states.

(~~bound states of heavy quark : much the same~~)

(Cooper pair : much the same ; different in details)

Consider

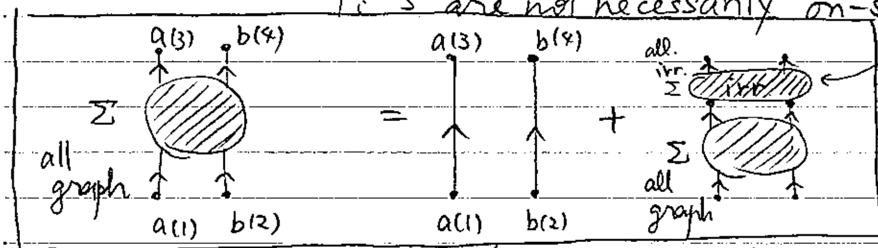
$$\iiiint \langle\Omega| T\{\psi_a(x_3)\psi_b(x_4)\psi_b^\dagger(x_2)\psi_a^\dagger(x_1)\}|\Omega\rangle\, e^{-ip_1\cdot x_1} e^{-ip_2\cdot x_2} e^{ip_3\cdot x_3} e^{ip_4\cdot x_4}\, d^4x_1\, d^4x_2\, d^4x_3\, d^4x_4$$

$$=: (2\pi)^4\delta^4(p_1+p_2-p_3-p_4)\ G(P_{CM}^\mu ; p^\mu, p'^\mu)$$

Relabel the momenta : $\begin{cases} p_1^\mu = p^\mu + \eta_a P_{CM}^\mu \\ p_2^\mu = -p^\mu + \eta_b P_{CM}^\mu \end{cases}$ $\begin{cases} p_3^\mu = p'^\mu + \eta_a P_{CM}^\mu \\ p_4^\mu = -p'^\mu + \eta_b P_{CM}^\mu . \end{cases}$

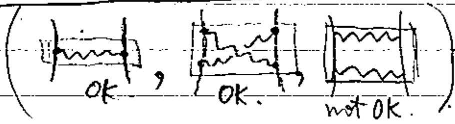
$\left(\eta_a = \frac{m_a}{m_a+m_b} \quad \eta_b = \frac{m_b}{m_a+m_b} . \right)$

p_i^μ's are not necessarily on-shell.



Two-particle irreducible graphs

- external legs are not included.
- remain connected when any one a-particle line and any one b-particle line are cut simultaneously.



$$\boxed{\begin{aligned} G(P_{CM}^\mu ; p^\mu, p'^\mu) &= (2\pi)^4\delta^4(p-p')\, D_a(P_{CM}, p')\, D_b(P_{CM}, p') \\ &+ D_a(P_{CM}, p')\, D_b(P_{CM}, p') \int \frac{d^4p''}{(2\pi)^4} K_{irr}(P_{CM}; p', p'')\, G(P_{CM}^\mu ; p^\mu, (p'-p'')^\mu) \end{aligned}}$$

(Bethe–Salpeter eq.)

non-rela parametrization

$$p^0 \Rightarrow \omega \qquad (p')^0 = \omega' \quad ; \quad P_{CM}^0 = (m_a+m_b)+(\Delta E).$$

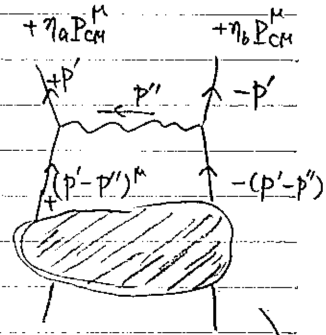
$$p_3'^{\mu} \Rightarrow \left(m_a + \eta_a(\Delta E) + \omega', \; \eta_a \vec{P}_{CM} + \vec{p}'\right)$$

$$p_4'^{\mu} \Rightarrow \left(m_b + \eta_b(\Delta E) - \omega', \; \eta_b \vec{P}_{CM} - \vec{p}'\right)$$

$$D_a^{(\text{Tree})} = \frac{i}{\left[\eta_a(\Delta E) + \omega' - \frac{(\eta_a \vec{P}_{CM} + \vec{p}')^2}{2m_a} + i\varepsilon\right]}$$

$$D_b^{(\text{tree})} = \frac{i}{\left[\eta_b(\Delta E) - \omega' - \frac{(\eta_b \vec{P}_{CM} - \vec{p}')^2}{2m_b} + i\varepsilon\right]}.$$

LO approximation to Kirr



in a photon exchange

$$K_{irr} = (-ieQ_a)(-ieQ_b)\frac{(-i)}{\left[(p'')^2 = (\omega'')^2 - (\vec{p}'')^2\right]}.$$

in a phonon exchange

$$\left(K_{irr} = \left(\frac{ig}{\Lambda}\right)^2 \frac{(+i)\;(\vec{p}''\cdot\vec{p}'')}{\left[(\omega'')^2 - v_s^2(\vec{p}'')^2 + i\varepsilon\right]} . \longleftarrow \mathcal{L}_{int} = \frac{g}{\Lambda}(\vec{\partial}\cdot\vec{\varphi})\,\psi^\dagger\psi \right)$$

Now, think of a case there are contributions of the form

$$G(P_{CM}^{\mu} ; p^{\mu}, p'^{\mu}) = \sum_n \left\{ \chi_n(p') \frac{i}{(P_{CM})^2 - M_n^2 + i\varepsilon} \chi_n^*(p) \right\} + \left(\text{non-pole terms}\right).$$

$\Longleftrightarrow \exists$ bound states

$$\boxed{\langle \Omega | T\{\psi_a(p_3)\psi_b(p_4)\} | n; \vec{P}_{CM*}\rangle = (2\pi)^4 \delta^3(\vec{P}_{CM} - \vec{P}_{CM*})\, \delta\left(m_a+m_b+(\Delta E) - E_{n,\vec{P}_{CM*}}\right)\cdot \chi_n(p')}$$

Q: Verify that

$$\left\{\begin{aligned} &[\langle\Omega|\psi(x)\psi(x)\psi^\dagger(x)\psi^\dagger(x)|\Omega\rangle] = +6 \\ &[G(P_{CM}^{\mu}; p^{\mu}, p'^{\mu})] = -6 \\ &[\chi_n(p')] = -2. \end{aligned}\right.$$

$$\left\{\begin{aligned} &[\langle\Omega|\psi(x)\psi(x)|\text{state}\rangle] = +2 \\ &[\langle\Omega|\psi(p)\psi(p)|\text{state}\rangle] = -6 \end{aligned}\right.$$

→ So, both the RHS & LHS of the eqn above have mass-dim -6 (sanity check)

Comparing the residue on a pole in the BS equation, we obtain

$$\chi_n(p') \cong D_a(\underline{\Delta E}, \vec{P}_{CM}; p')\, D_b(\underline{\Delta E}, \vec{P}_{CM}; p') \times \int \frac{d^3\vec{p}''}{(2\pi)^3} \int \frac{d\omega''}{2\pi} \frac{i(eQ_a eQ_b)}{(\omega'')^2 - |\vec{p}''|^2}\, \chi_n(p'-p'') \qquad \text{——} (*)$$

$$\boxed{\text{Suppose that we can ignore } (\omega'')^2 \text{ against } |\vec{p}''|^2 \text{ in the dominant region of the integral (verified later).}} \quad \text{——} (**)$$

Then we see that the $[\omega' = (p')^0]$-dependence in the RHS of $(*)$ comes only from $D_a \cdot D_b$. So,

$$\chi_n(\omega', \vec{p}'; \vec{P}_{CM}) = \frac{i}{\left[\eta_a(\underline{\Delta E})_n + \omega' - \frac{(\eta_a \vec{P}_{CM} + \vec{p}')^2}{2m_a} + i\varepsilon\right]} \frac{i \times \underline{\chi}_n(\vec{p}'; \vec{P}_{CM})}{\left[\eta_b(\underline{\Delta E})_n - \omega' - \frac{(\eta_b \vec{P}_{CM} - \vec{p}')^2}{2m_b} + i\varepsilon\right]},$$

$$\psi_n(\vec{p}'; \vec{P}_{CM}) := \int \frac{d\omega'}{2\pi} \chi_n(\omega', \vec{p}'; \vec{P}_{CM})$$

$$= \frac{-\underline{\chi}_n(\vec{p}'; \vec{P}_{CM})}{2\pi} \frac{1}{\left[(\underline{\Delta E})_n - \frac{\vec{P}_{CM}^2}{2(m_a+m_b)} - \frac{|\vec{p}'|^2}{2\mu_{ab}}\right]} \int d\omega' \left(\frac{1}{\left[\eta_a \underline{\Delta E}_n + \omega' - \frac{(\ )^2}{2m_a} + i\varepsilon\right]} + \frac{1}{\left[\eta_b (\underline{\Delta E})_n - \omega' - \frac{(\ )^2}{2m_b} + i\varepsilon\right]} \right)$$

$$= \frac{-(-2\pi i)}{2\pi} \frac{\underline{\chi}_n(\vec{p}'; \vec{P}_{CM})}{[\quad]} = \frac{(+i)}{\left[(\underline{\Delta E})_n - \frac{\vec{P}_{CM}^2}{2(m_a+m_b)} - \frac{\vec{p}'^2}{2\mu_{ab}}\right]} \underline{\chi}_n(\vec{p}', \vec{P}_{CM}).$$

Thus, the eqn $(*)$ can be rewritten as

$$\int \frac{d^3\vec{p}''}{(2\pi)^3} \frac{(eQ_a eQ_b)}{|\vec{p}''|^2} \psi_n(\vec{p}' - \vec{p}''; \vec{P}_{CM}) \overset{(*)}{\cong} \frac{i}{D_a D_b} \chi_n(p') = i\,\underline{\chi}_n = \left(\underline{\Delta E}_n - \frac{\vec{P}_{CM}^2}{2(m_a+m_b)} - \frac{(\vec{p}')^2}{2\mu_{ab}}\right) \psi_n(\vec{p}'; \vec{P}_{CM})$$

$$\Big\Downarrow \text{ Fourier transform in } \vec{p}' \to \vec{r}$$

$$\boxed{\left((\underline{\Delta E})_n - \frac{\vec{P}_{CM}^2}{2(m_a+m_b)}\right) \tilde{\psi}_n(\vec{r}; \vec{P}_{CM}) = \left(\frac{-\vec{\partial}_r^2}{2\mu_{ab}} + \frac{e^2 Q_a Q_b}{4\pi r}\right) \tilde{\psi}_n(\vec{r}; \vec{P}_{CM})}$$

Schrödinger equation

$$\left(\frac{1}{\mu_{ab}} := \frac{1}{m_a} + \frac{1}{m_b} = \left(\frac{m_a m_b}{(m_a+m_b)}\right)^{-1} \text{ reduced mass}\right)$$

$$\boxed{\chi_n(p') \text{ "is" the matrix element } \langle \Omega | T\{\psi_a \psi_b\} | \text{bound state}_n \rangle}$$