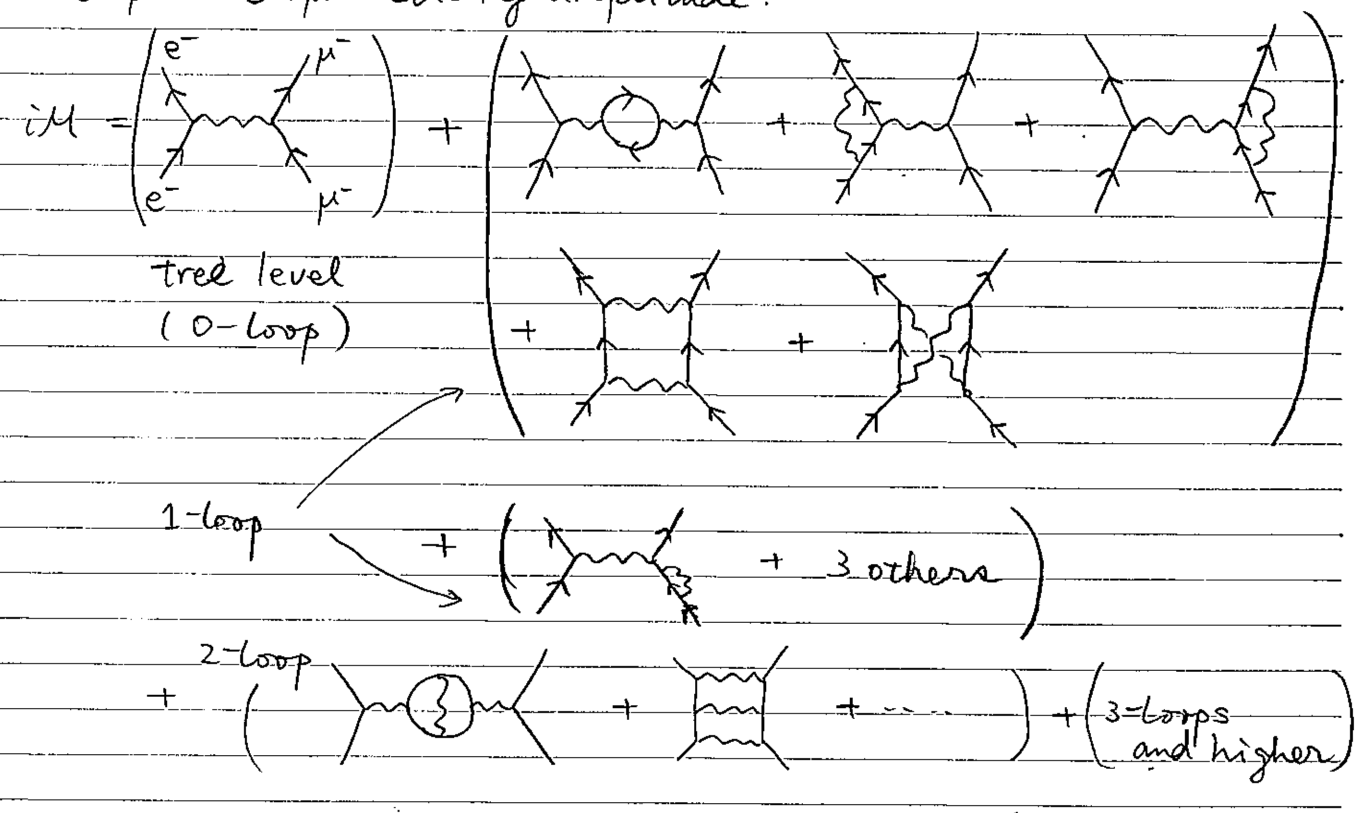
§5 Introduction to 1-loop computations

In perturbative calculations, contributions to a given correlation function / scattering amplitude are sorted out in the order of the # of loops.

ex $e^- + \mu^- \to e^- + \mu^-$ scattering amplitude.



Time evolution of a state may involve creation / annihilation of e^+e^-, $\mu^+\mu^-$, γ virtually in the intermediate processes. multiple times.

Scattering amplitudes are therefore in the form of

$$i\mathcal{M} = e^2 \times (\text{kinematics}) + e^4 \times (\text{kinematics}) + e^6 \times (\text{kinematics}) + \cdots$$

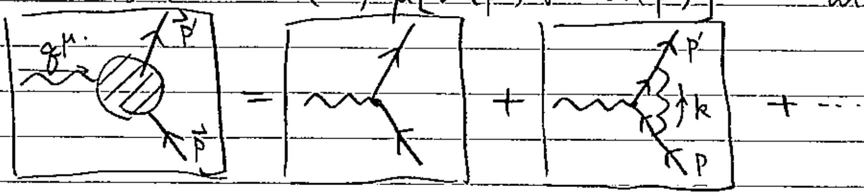
☆ 1-loop computations (and maybe higher loop) are necessary when high precision is required.

☆ Some processes are absent at tree level, and generated at higher-loop for the first time. (eg: )

an example: anomalous magnetic moment of μ^-

The three point amplitude ($\gamma^{(*)}+\mu^- \to \mu^-$) only has

$i\mathcal{M}^{\text{tree}} = i(-Qe)\,\varepsilon_\mu\left[\bar{u}(\vec{p}\,')\,\gamma^\mu\, u(\vec{p})\right]$ at the tree level.



$\left[p'_\mu = p_\mu + q_\mu\right]$

(interested in the $q^2=0$ limit.)

At the 1-Loop level,

$$i\mathcal{M}^{(1\text{-Loop})} = \int \bar{u}(\vec{p}\,')\left[i(-Q_\mu e)\gamma^\kappa\right]\frac{i\left[(\not{p}'-\not{k})+M\right]}{\left((p'-k)^2-M^2+i\varepsilon\right)}\left[i(-Q_\mu e)\gamma^\mu\right]\frac{i\left[(\not{p}-\not{k})+M\right]}{\left((p-k)^2-M^2+i\varepsilon\right)}$$

$$\left[i(-Q_\mu e)\gamma^\lambda\right] u(\vec{p}) \times \left(\frac{-i\eta_{\kappa\lambda}}{k^2+i\varepsilon}\right) \times \varepsilon_\mu(q)\,\frac{d^4k}{(2\pi)^4}$$

$$=(-i)\int\frac{d^4k}{(2\pi)^4}\left[i(-Qe)\right](Qe)^2\frac{\bar{u}(\vec{p}\,')\gamma^\kappa\left[(\not{p}'-\not{k})+M\right]\not{\varepsilon}\left[(\not{p}-\not{k})+M\right]\gamma_\kappa u(\vec{p})}{\left[k^2+i\varepsilon\right]\left[(p'-k)^2-M^2+i\varepsilon\right]\left[(p-k)^2-M^2+i\varepsilon\right]}.$$

A trick that makes integration easy:

$$\frac{1}{A\,B} = \int_0^1 dx \frac{1}{[xA+(1-x)B]^2} \qquad (\text{RHS}) = \int_0^1 dx \frac{1}{(A-B)^2\left(x+\frac{B}{A-B}\right)^2} = \frac{1}{(A-B)^2}\left(\frac{A-B}{B} - \frac{A-B}{A}\right)$$

$$\frac{1}{ABC} = \int_{\triangle} dx\,dy \frac{2}{[xA+yB+(1-x-y)C]^3}$$

$$(\text{RHS}) = \int_0^1 dx \int_0^x dy \frac{2}{(B-C)^3\left[y+\frac{xA+(1-x)C}{(B-C)}\right]^3} = \int_0^1 dx \frac{1}{(B-C)}\left(\frac{1}{[xA+(1-x)C]^2} - \frac{1}{[xA+(1-x)B]^2}\right)$$

$$= \frac{1}{(B-C)}\left(\frac{1}{AC} - \frac{1}{AB}\right) = \frac{1}{ABC}$$

$$\boxed{\frac{1}{A_1A_2\cdots A_n} = \int_{0\le x_i} d^n x \frac{\delta(\Sigma_i x_i - 1)\,(n-1)!}{[x_1A_1+x_2A_2+\cdots+x_nA_n]^n}}$$

by induction $(\text{RHS})_n = \frac{1}{(A_{n-1}-A_n)}\left\{(\text{RHS})_{n-1}(A_1,\cdots,\check{A}_{n-1},A_n) - (\text{RHS})_{n-1}(A_1,\cdots,A_{n-1},\check{A}_n)\right\}$

$$= \frac{1}{(A_{n-1}-A_n)} \times \frac{1}{(A_1A_2\cdots A_{n-2})}\left(\frac{1}{A_n} - \frac{1}{A_{n-1}}\right) = (\text{LHS})_n \;//$$

If necessary

$$\frac{1}{A^2BC} = -\frac{\partial}{\partial A}\left(\frac{1}{ABC}\right) = \int_{\triangle} dx\,dy \frac{2\cdot 3\, x}{[xA+yB+(1-x-y)C]^4}$$

etc.

So, in particular

$$\frac{1}{(k^2+i\varepsilon)}\,\frac{1}{[(p'-k)^2-M^2+i\varepsilon]}\,\frac{1}{[(p-k)^2-M^2+i\varepsilon]}$$

$$= \int_{\triangle} dx\,dy \frac{2}{[x(p'-k)^2+y(p-k)^2+(1-x-y)k^2-(x+y)M^2+i\varepsilon]^3}$$

used $(p')^2 = M^2 = p^2$ on-shell

$$= \int_{\triangle} dx\,dy \frac{2}{[k^2-2k\cdot(xp'+yp)+i\varepsilon]^3}$$

$(xp'+yp)^2 = x^2M^2+y^2M^2+2xy\,p'\cdot p$

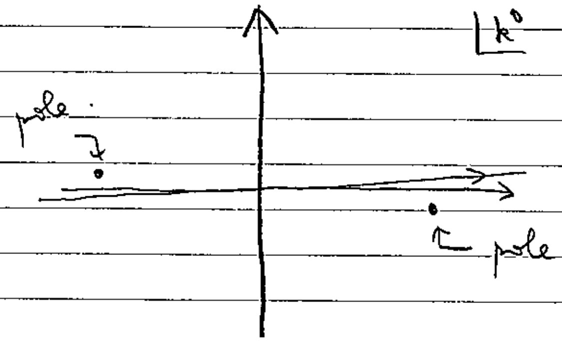
$q^2 = (p'-p)^2 = 2M^2-2p'\cdot p$

$$= \int_{\triangle} dx\,dy \frac{2}{[(k')^2-((x+y)^2M^2-xy\,q^2)+i\varepsilon]^3}$$

$k' := k-(xp'+yp)$ to complete a square.

Now, we think of carrying out $\frac{d^4k}{(2\pi)^4}$ before $dxdy$.

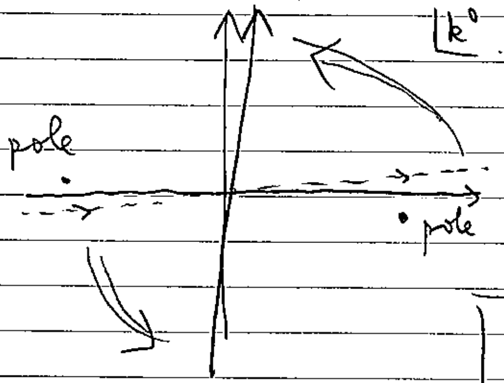
The k^0-integration, in particular, has the following structure:



$$\frac{dk^0}{2\pi}\frac{(\text{numerator})}{\left[(k'^0)^2-\{(\vec{k}')^2+\underbrace{(x+y)^2M^2}+\underbrace{xy}\underbrace{(-q^2)}\}+i\varepsilon\right]^3}$$

pos. pos. pos. pos. if q is spacelike

So, it is possible to rotate the contour of integration into



[called Wick rotation]

The new integration is parametrized by $k'^0 = i k'^4$ (so $k^4 \in \mathbb{R}$)

$$\boxed{-i\int_{-\infty}^{+\infty}\frac{dk'^0}{(2\pi)}\int_{\mathbb{R}^3}\frac{d^3k'}{(2\pi)^3} \Rightarrow +\int_{\mathbb{R}^4}\frac{d^4k'_E}{(2\pi)^4}}$$

$$\boxed{\left[(k')^2-\{(x+y)^2M^2+xy(-q^2)\}+i\varepsilon\right]^3 \Longrightarrow -\left[k_E^2+\{(x+y)^2M^2+xy(-q^2)\}\right]^3}$$

The numerator

- $(M^2)\,\gamma^\kappa \not{\varepsilon}\gamma_\kappa = -2M^2\not{\varepsilon}$ $\quad\left(\text{use } \gamma^\kappa \not{A}\gamma_\kappa = -2\not{A}\right)$

- $M\left\{\gamma^\kappa(\not{p}'-\not{k})\not{\varepsilon}\gamma_\kappa + \gamma^\kappa\not{\varepsilon}(\not{p}-\not{k})\gamma_\kappa\right\}$ $\quad\left(\begin{aligned}\text{use } \gamma^\kappa\not{A}\not{B}\gamma_\kappa &= 2\not{B}\gamma_\kappa A^\kappa - \not{A}\gamma^\kappa\not{B}\gamma_\kappa \\ &= 2(\not{B}\not{A}+\not{A}\not{B}) = 4(A\cdot B)\end{aligned}\right)$

$$= 4M\,\varepsilon_\mu(p'+p-2k)^\mu$$

$$= 4M\,\varepsilon_\mu\left((1-2x)p'+(1-2y)p-2k'\right)^\mu \qquad \left(k'_\mu = k_\mu-(xp'+yp)_\mu\right)$$

- $\gamma^\kappa(\not{p}'-\not{k})\not{\varepsilon}(\not{p}-\not{k})\gamma_\kappa$

$$= -2\,\not{[(1-y)p-xp'-k']}\,\not{\varepsilon}\,\not{[(1-x)p'-yp-k']}. \quad \llcorner(**)$$

use
$$\begin{aligned}\gamma^\kappa\not{A}\not{B}\not{C}\gamma_\kappa &= \not{B}\not{C}\gamma_\kappa 2A^\kappa - \not{A}\not{C}\gamma_\kappa 2B^\kappa + \not{A}\not{B}\gamma^\kappa\not{C}\gamma_\kappa \\ &= 2(\not{B}\not{C}\not{A} - \not{A}\not{C}\not{B} - \not{A}\not{B}\not{C}). \\ &= \underline{4(B\cdot C)\not{A}} - 2\not{C}\not{B}\not{A} - \underline{4\not{A}(B\cdot C)}\end{aligned}$$

☆ drop all the terms with an odd power of k' $\left(\text{integration of an odd function} = 0\right)$

☆ replace $k'^\rho k'^\sigma$ by $\frac{1}{\text{"}4\text{"}}\eta^{\rho\sigma}(k')^2$ $\quad\left(\begin{aligned}&\text{"}4\text{" is the spacetime dimension} \\ &\eta_{\rho\sigma}\eta^{\rho\sigma} = \text{"}4\text{"}\end{aligned}\right)$

Then

$$(**) = \frac{-2(k')^2}{4}\left[-2\not{\varepsilon}\right] - 2\left\{xy\,\not{p}'\not{\varepsilon}\not{p} - x(1-x)\not{p}'\not{\varepsilon}\not{p}' - y(1-y)\not{p}\not{\varepsilon}\not{p} + (1-x)(1-y)\not{p}\not{\varepsilon}\not{p}'\right\}$$

To evaluate the 2nd term, we exploit

$$\not{p}\,u(\vec{p}) = M u(\vec{p}) \quad\text{and}\quad \bar{u}(\vec{p}')\not{p}' = \bar{u}(\vec{p}')M.$$

So, when sandwitched between $\bar{u}(\vec{p}')$ and $u(\vec{p})$

$$\begin{aligned}(**) = (k')^2\not{\varepsilon} - 2M^2\not{\varepsilon}\left\{xy + x(1-x) + y(1-y) - (1-x)(1-y)\right\} + 2\cdot\overbrace{(2p\cdot p')}^{2M^2-q^2}(1-x)(1-y)\not{\varepsilon} \\ -2\left\{-x(1-x)M\,2(\varepsilon\cdot p') - y(1-y)\,2(\varepsilon\cdot p)M + (1-x)(1-y)\left(2(\varepsilon\cdot p)+2(\varepsilon\cdot p')\right)M\right\}\end{aligned}$$

So, the numerator becomes

$$\begin{aligned}&\left[\bar{u}(\vec{p}')\not{\varepsilon}u(\vec{p})\right]\times\Big(-2M^2\underbrace{\left\{xy + x(1-x) + y(1-y) - 3(1-x)(1-y) + 1\right\}}_{\to\{-2+4(x+y)-(x+y)^2\}} - (k_E^2) - 2(1-x)(1-y)q^2\Big) \\ &+\left[\bar{u}(\vec{p}')u(\vec{p})\right]\times(-4M)\;\varepsilon\cdot(p+p')\Big\{\underbrace{(1-x)(1-y) - \frac{x(1-x)+y(1-y)}{2}}_{=\frac{(1-x-y)(2-x-y)}{2}} - (1-x-y)\Big\} \to = (1-x-y)\frac{(x+y)}{2}\end{aligned}$$

Let us parametrize

$$i\mathcal{M} =: i(-Qe)\,\bar{u}(\vec{p}')\left(\varepsilon_\mu \gamma^\mu F_1(q^2) - \frac{F_2(q^2)}{4M}\varepsilon_\mu[\gamma^\mu, \gamma^\nu] q_\nu\right) u(\vec{p})$$

$$= i(-Qe)\,\bar{u}(\vec{p}')\left(\varepsilon_\mu \gamma^\mu (F_1 + F_2) - \varepsilon_\mu (p'+p)^\mu \frac{F_2}{2M}\right) u(\vec{p})$$

using $\not{p}\, u(\vec{p}) = M u(\vec{p})$ and $\bar{u}(\vec{p}')\not{p}' = \bar{u}(\vec{p}')M$.

At tree-level, $F_1 = 1$ and $F_2 = 0$.

F_1 : Dirac's form factor
F_2 : Pauli's form factor

are non-trivial functions of q^2 at the quantum level.

(μ^I : regarded "elementary" in QED though)

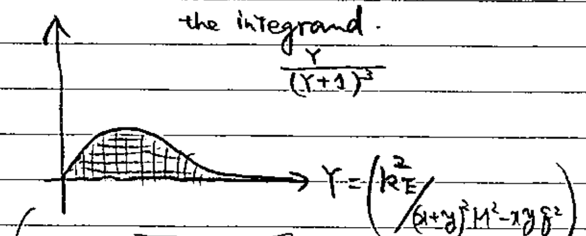
$$F_2^{(1\text{-loop})} = -2M \int \frac{d^4k_E}{(2\pi)^4} \int_\Delta dx\,dy\, \frac{2(Qe)^2 \; 2M \; (1-x-y)(x+y)}{-\left[(k_E^2) + (x+y)^2 M^2 + xy(-q^2)\right]^3}$$

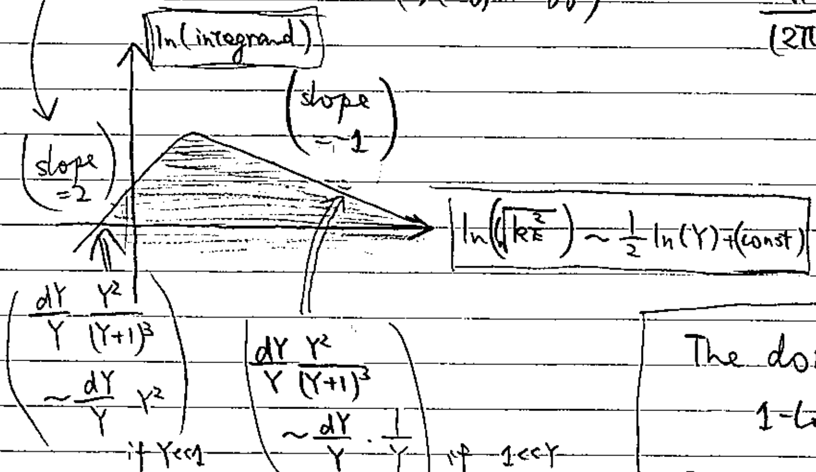
Now we carry out d^4k_E integral

$$F_2^{(1)} = \int_\triangle dx\,dy\, \frac{(2M)^2(Qe)^2}{[(x+y)^2M^2+xy(-q^2)]^3}\left\{\frac{2(1-x-y)}{(x+y)}\right\} \underbrace{\int \frac{d^4k_E}{(2\pi)^4}\frac{1}{\left[k_E^2/((x+y)^2M^2-xyq^2)+1\right]^3}}$$

$$= \frac{\boxed{2\pi^2 = \mathrm{vol}(S^3)}}{(2\pi)^4}\,\frac{1}{2}\int_0^{+\infty} d(k_E^2)\,\frac{(k_E^2)}{\left[\left(k_E^2/((x+y)^2M^2-xyq^2)\right)+1\right]^3}$$

$$= \frac{\pi^2}{(2\pi)^4}\left[(x+y)^2M^2-xyq^2\right]^2 \cdot \underbrace{\int_0^{+\infty} dY\,\frac{Y}{[Y+1]^3}}_{=\,1/2}$$





The dominant contribution ~~to~~ the 1-loop integral for $F_2^{(1)}$ is from $k_E \sim O(M^2)$ or $O(-q^2)$.

So $$F_2^{(1)} = \int_\triangle dx\,dy\, \frac{(Qe)^2}{(4\pi)^2}\,\frac{\frac{1}{2}(2M)^2\{2(1-x-y)(x+y)\}}{[(x+y)^2M^2-xyq^2]}$$

$(x,y)\to(x,z)$, $z=(x+y)$

$$F_2^{(1)}(q^2=0) = \int_0^1 dz\,\frac{(Qe)^2}{(4\pi)^2}\,\frac{(2M)^2}{z^2M^2}(1-z)z\times\left(z\leftarrow\int_0^z dx\right)$$

$$= \frac{\alpha_e}{\pi}\times\frac{1}{2}.$$

The anomalous magnetic moment:

$$\boxed{g = 2+2F_2(q^2=0) \simeq 2+\frac{\alpha_e}{\pi}} \quad \text{at 1-loop order.}$$

$$\alpha_e \simeq 1/137$$

It is also possible to compute $\left.\frac{\partial F_2}{\partial q^2}\right|_{q^2=0}$ and higher order.
$\approx$ quantum "radius^2" of μ.