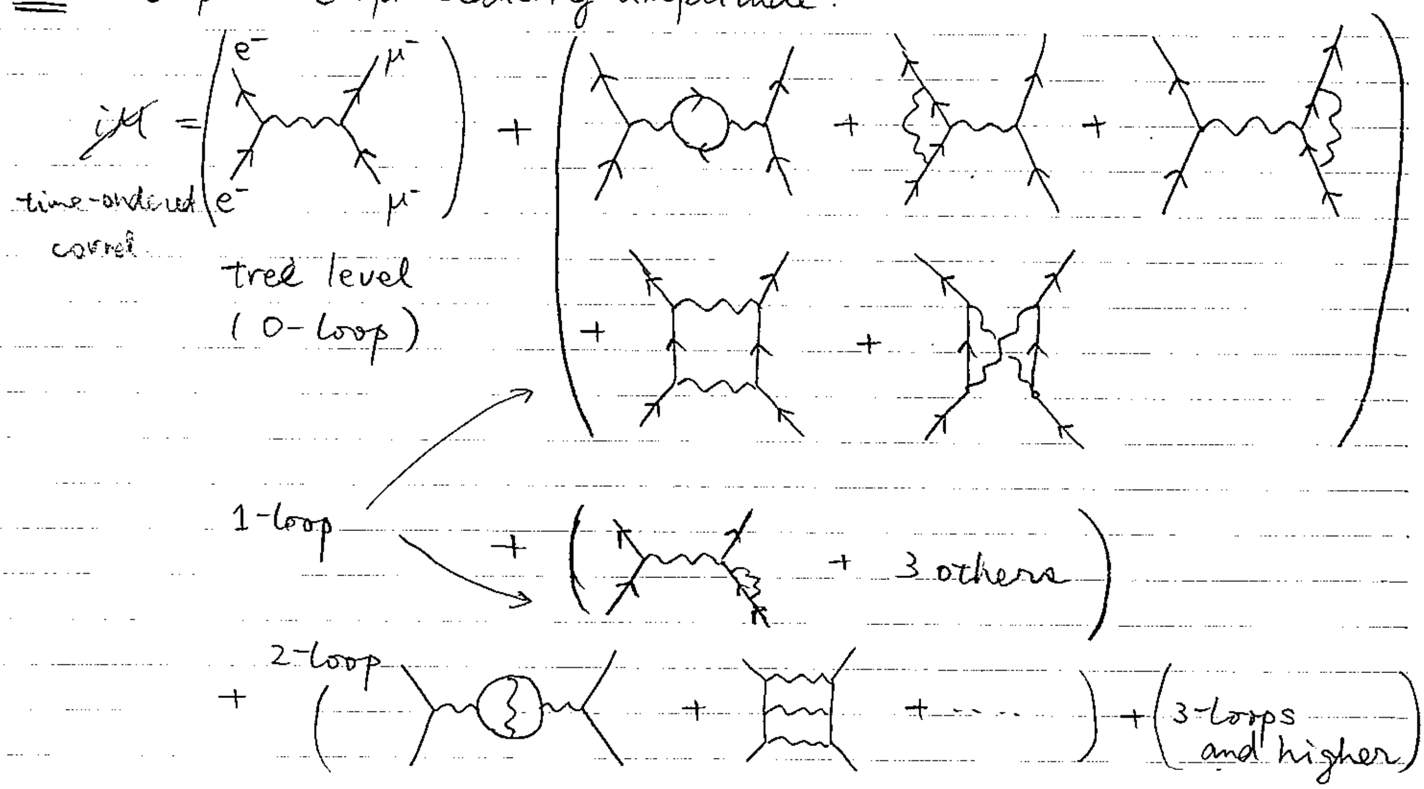
§5 Introduction to 1-loop computations

In perturbative calculations, contributions to a given correlation function / scattering amplitude are sorted out in the order of the # of loops.

ex $e^- + \mu^- \to e^- + \mu^-$ scattering amplitude.



Time evolution of a state may involve creation / annihilation of e^+e^-, $\mu^+\mu^-$, γ virtually in the intermediate processes. multiple times.

Scattering amplitudes are therefore in the form of

$$i\mathcal{M} = e^2 \times (\text{kinematics}) + e^4 \times (\text{kinematics}) + e^6 \times (\text{kinematics}) + \cdots$$

☆ 1-loop computations (and maybe higher loop) are necessary when high precision is required.

☆ Some processes are absent at tree level, and generated at higher-loop for the first time. (eg: 

)

an example: anomalous magnetic moment of μ^-

The three point amplitude ($\gamma^{(*)} + \mu^- \to \mu^-$) only has

$$i\mathcal{M}^{\text{tree}} = i(-Qe)\,\varepsilon_\mu \left[\bar{u}(\vec{p}')\,\gamma^\mu\, u(\vec{p})\right]$$ at the tree level.

$+ \cdots$

$[p'_\mu = p_\mu + q_\mu]$

(interested in the $q^2 = 0$ limit.)

At the 1-Loop level,

$$\boxed{i\mathcal{M}^{(1\text{-loop})}_{(\triangle)} = \int \bar{u}(\vec{p}')\left[i(-Q_\mu e)\gamma^\kappa\right] \frac{i\left[(\cancel{p}'-\cancel{k})+M\right]}{((p'-k)^2-M^2+i\varepsilon)}\left[i(-Q_\mu e)\gamma^\mu\right]\frac{i\left[(\cancel{p}-\cancel{k})+M\right]}{((p-k)^2-M^2+i\varepsilon)} \left[i(-Q_\mu e)\gamma^\lambda\right] u(\vec{p}) \times \left(\frac{-i\eta_{\kappa\lambda}}{k^2+i\varepsilon}\right) \times \varepsilon_\mu(q)\,\frac{d^4k}{(2\pi)^4}}$$

(wavefn くりこみが ビミョー)

$$= (-i)\int \frac{d^4k}{(2\pi)^4}\left[i(-Qe)\right](Qe)^2 \frac{\bar{u}(\vec{p}')\,\gamma^\kappa\left[(\cancel{p}'-\cancel{k})+M\right]\cancel{\varepsilon}\left[(\cancel{p}-\cancel{k})+M\right]\gamma_\kappa\, u(\vec{p})}{\left[k^2+i\varepsilon\right]\left[(p'-k)^2-M^2+i\varepsilon\right]\left[(p-k)^2-M^2+i\varepsilon\right]}.$$

A trick that makes integration easy:

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[xA+(1-x)B]^2}$$

$\because$ $(\mathrm{RHS}) = \int_0^1 dx \frac{1}{(A-B)^2 \left(x+\frac{B}{A-B}\right)^2} = \frac{1}{(A-B)^2}\left(\frac{A-B}{B} - \frac{A-B}{A}\right)$

$$\frac{1}{ABC} = \int_{\triangle} dx\, dy \frac{2}{[xA+yB+(1-x-y)C]^3}$$

$\because$ $(\mathrm{RHS}) = \int_0^1 dx \int_0^x dy \frac{2}{(B-C)^3 \left[y + \frac{xA+(1-x)C}{(B-C)}\right]^3} = \int_0^1 dx \frac{1}{(B-C)}\left(\frac{1}{[xA+(1-x)C]^2} - \frac{1}{[xA+(1-x)B]^2}\right)$

$$= \frac{1}{(B-C)}\left(\frac{1}{AC} - \frac{1}{AB}\right) = \frac{1}{ABC}$$

$$\boxed{\frac{1}{A_1 A_2 \cdots A_n} = \int_{0\le x_i} d^n x \frac{\delta(\Sigma_i x_i - 1)\,(n-1)!}{[x_1A_1 + x_2A_2 + \cdots + x_nA_n]^n}}$$

by induction $(\mathrm{RHS})_n = \frac{1}{(A_{n-1}-A_n)}\left\{(\mathrm{RHS})_{n-1}(A_1,\cdots,\check{A}_{n-1},A_n) - (\mathrm{RHS})_{n-1}(A_1,\cdots,A_{n-1},\check{A}_n)\right\}$

$$= \frac{1}{(A_{n-1}-A_n)} \times \frac{1}{(A_1A_2\cdots A_{n-2})}\left(\frac{1}{A_n} - \frac{1}{A_{n-1}}\right) = (\mathrm{LHS})_n \quad //$$

If necessary

$$\frac{1}{A^2BC} = -\frac{\partial}{\partial A}\left(\frac{1}{ABC}\right) = \int_{\triangle} dx\, dy \frac{2\cdot 3\, x}{[xA+yB+(1-x-y)C]^4}$$

etc.

So, in particular

$$\frac{1}{(k^2+i\varepsilon)}\frac{1}{[(p'-k)^2-M^2+i\varepsilon]}\frac{1}{[(p-k)^2-M^2+i\varepsilon]}$$

$$= \int_{\triangle} dx\, dy \frac{2}{[x(p'-k)^2 + y(p-k)^2 + (1-x-y)k^2 - (x+y)M^2 + i\varepsilon]^3}$$

used $(p')^2 = M^2 = p^2$ on-shell

$$= \int_{\triangle} dx\, dy \frac{2}{[k^2 - 2k\cdot(xp'+yp) + i\varepsilon]^3}$$

$$= \int_{\triangle} dx\, dy \frac{2}{[(k')^2 - ((x+y)^2M^2 - xy\,q^2) + i\varepsilon]^3}$$

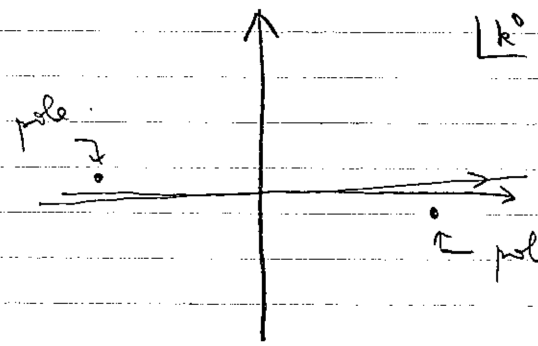
$(xp'+yp)^2 = x^2M^2 + y^2M^2 + 2xy\,p'\cdot p$

$q^2 = (p'-p)^2 = 2M^2 - 2p'\cdot p$

$k' := k - (xp'+yp)$ to complete a square.

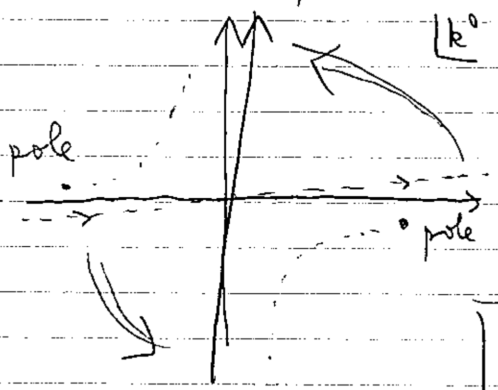
Now, we think of carrying out $\frac{d^4k}{(2\pi)^4}$ before $dx\,dy$.

The k^0-integration, in particular, has the following structure:



$$\frac{dk^0}{2\pi}\,\frac{(\text{numerator})}{\left[(k^0)^2-\{\underset{\uparrow\ \text{pos.}}{(\vec{k})^2}+\underset{\uparrow\ \text{pos.}}{\underbrace{(x+y)^2M^2}}+\underset{\uparrow\ \text{pos.}}{\underbrace{xy}}\underset{\uparrow\ \text{pos. if } q \text{ is spacelike}}{\underbrace{(-q^2)}}\}+i\varepsilon\right]^3}$$

So, it is possible to rotate the contour of integration into



[called Wick rotation]

The new integration is parametrized by $k'^0 = i\,k'^4$ (so $k^4 \in \mathbb{R}$)

$$\boxed{-i\int_{-\infty}^{+\infty}\frac{dk'^0}{(2\pi)}\int_{\mathbb{R}^3}\frac{d^3k'}{(2\pi)^3} \Rightarrow +\int_{\mathbb{R}^4}\frac{d^4k'_E}{(2\pi)^4}}$$

$$\boxed{\left[(k')^2-\{(x+y)^2M^2+xy(-q^2)\}+i\varepsilon\right]^3 \Rightarrow -\left[k_E^2+\{(x+y)^2M^2+xy(-q^2)\}\right]^3}$$

The numerator

- $(M^2)\,\gamma^\kappa \not{\varepsilon}\,\gamma_\kappa = -2M^2 \not{\varepsilon}$ $\qquad$ (use $\gamma^\kappa \not{A}\,\gamma_\kappa = -2\not{A}$)

- $M\left\{\gamma^\kappa \not{(p'-k)}\not{\varepsilon}\,\gamma_\kappa + \gamma^\kappa \not{\varepsilon}\not{(p-k)}\,\gamma_\kappa\right\}$ $\qquad$ (use $\gamma^\kappa \not{A}\not{B}\,\gamma_\kappa = 2\not{B}\,\gamma_\kappa A^\kappa - \not{A}\,\gamma^\kappa \not{B}\,\gamma_\kappa = 2(\not{B}\not{A} + \not{A}\not{B}) = 4(A\cdot B)$)

$$= 4M\ \varepsilon_\mu (p'+p-2k)^\mu$$

$$= 4M\ \varepsilon_\mu \left((1-2x)p' + (1-2y)p - 2k'\right)^\mu \qquad \left(k'_\mu = k_\mu - (xp'+yp)_\mu\right)$$

- $\gamma^\kappa \not{(p'-k)}\not{\varepsilon}\not{(p-k)}\,\gamma_\kappa$

$$= -2\,\underbrace{\left[\not{((1-y)p - xp' - k')}\right]\not{\varepsilon}\left[\not{((1-x)p' - yp - k')}\right]}_{(**)}.$$

use
$$\gamma^\kappa \not{A}\not{B}\not{C}\,\gamma_\kappa = \not{B}\not{C}\,\gamma_\kappa 2A^\kappa - \not{A}\not{C}\,\gamma_\kappa 2B^\kappa + \not{A}\not{B}\,\gamma^\kappa \not{C}\,\gamma_\kappa$$
$$= 2(\not{B}\not{C}\not{A} - \not{A}\not{C}\not{B} - \not{A}\not{B}\not{C})$$
$$= \underline{4(B\cdot C)\not{A}} - 2\not{C}\not{B}\not{A} - \underline{4\not{A}(B\cdot C)}$$
cancel

☆ drop all the terms with an odd power of k' (integration of an odd function $= 0$)

☆ replace $k'^\rho k'^\sigma$ by $\frac{1}{\text{“}4\text{”}}\eta^{\rho\sigma}(k')^2$ $\qquad$ (“4” is the spacetime dimension $\eta_{\rho\sigma}\eta^{\rho\sigma} = $ “4”)

Then
$$(**) = \frac{-2(k')^2}{4}[-2\not{\varepsilon}] - 2\left\{xy\not{p}'\not{\varepsilon}\not{p} - x(1-x)\not{p}'\not{\varepsilon}\not{p}' - y(1-y)\not{p}\not{\varepsilon}\not{p} + (1-x)(1-y)\not{p}\not{\varepsilon}\not{p}'\right\}$$

To evaluate the 2nd term, we exploit

$$\not{p}\,u(\vec{p}) = M u(\vec{p}) \quad \text{and} \quad \bar{u}(\vec{p}')\not{p}' = \bar{u}(\vec{p}')M.$$

So, when sandwiched between $\bar{u}(\vec{p}')$ and $u(\vec{p})$

$$(**) = (k')^2\not{\varepsilon} - 2M^2\not{\varepsilon}\left\{xy + x(1-x) + y(1-y) - (1-x)(1-y)\right\} + 2\cdot\overbrace{(2p\cdot p')}^{2M^2 - q^2}(1-x)(1-y)\not{\varepsilon}$$
$$-2\left\{-x(1-x)M\,2(\varepsilon\cdot p') - y(1-y)\,2(\varepsilon\cdot p)M + (1-x)(1-y)\left(2(\varepsilon\cdot p) + 2(\varepsilon\cdot p')\right)M\right\}$$

So, the numerator becomes

$$\left[\bar{u}(\vec{p}')\not{\varepsilon}\,u(\vec{p})\right]\times\left(-2M^2\overbrace{\left\{xy + x(1-x) + y(1-y) - 3(1-x)(1-y) + 1\right\}}^{=\{-2+4(x+y)-(x+y)^2\}} - (k_E^2) - 2(1-x)(1-y)q^2\right)$$

$$+\left[\bar{u}(\vec{p}')u(\vec{p})\right]\times(-4M)\,\varepsilon\cdot(p+p')\left\{\underbrace{(1-x)(1-y) - \frac{x(1-x)+y(1-y)}{2}}_{=\frac{(1-x-y)(2-x-y)}{2}} - (1-x-y)\right\}$$

$$= (1-x-y)\frac{(x+y)}{2}$$

Let us parametrize

$$i\mathcal{M} =: i(-Qe)\,\bar{u}(\vec{p}\,')\left(\varepsilon_\mu \gamma^\mu F_1(q^2) - \frac{F_2(q^2)}{4M}\varepsilon_\mu[\gamma^\mu, \gamma^\nu]\, q_\nu\right) u(\vec{p})$$

$$= i(-Qe)\,\bar{u}(\vec{p}\,')\left(\varepsilon_\mu \gamma^\mu (F_1 + F_2) - \varepsilon_\mu (p'+p)^\mu \frac{F_2}{2M}\right) u(\vec{p})$$

using $\not{p}\, u(\vec{p}) = M u(\vec{p})$ and $\bar{u}(\vec{p}\,')\not{p}' = \bar{u}(\vec{p}\,')M$.

At tree-level, $F_1 = 1$ and $F_2 = 0$.

F_1 : Dirac's form factor
F_2 : Pauli's form factor

are non-trivial functions of q^2 at the quantum level.

($\mu^\pm$: regarded "elementary" in QED though)

($p^\mp$: $g = 2F_2 \neq 2$.)

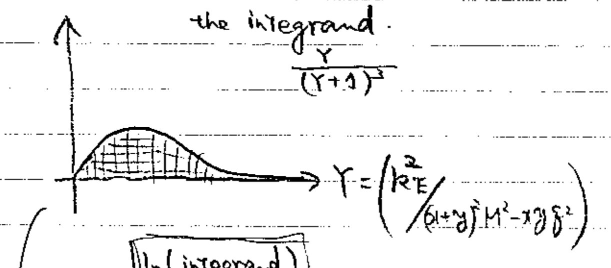
$$F_2^{(1\text{-loop})} = -2M \int \frac{d^4k_E}{(2\pi)^4} \int_\Delta dx\,dy\, \frac{2(Qe)^2\; 2M\; (1-x-y)(x+y)}{-[(k_E^2) + (x+y)^2 M^2 + xy(-q^2)]^3}$$

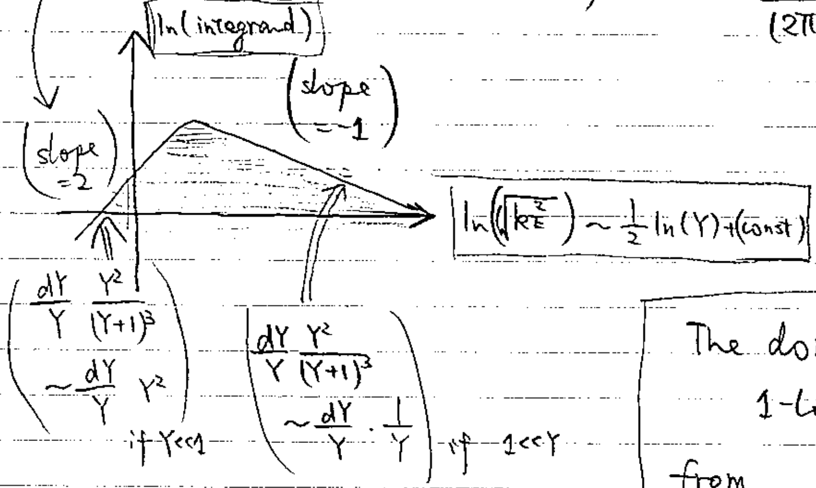
Now we carry out d^4k_E integral

$$F_2^{(1)} = \int_{\Delta} dx\, dy \frac{(2M)^2(Qe)^2}{[(x+y)^2M^2 + xy(-q^2)]^3} \left\{\frac{2(1-x-y)}{(x+y)}\right\} \underbrace{\int \frac{d^4k_E}{(2\pi)^4} \frac{1}{\left[k_E^2/((x+y)^2M^2 - xyq^2) + 1\right]^3}}$$

$$= \frac{\boxed{2\pi^2 = \mathrm{vol}(S^3)}}{(2\pi)^4} \frac{1}{2}\int_0^{+\infty} d(k_E^2) \frac{(k_E^2)}{\left[\left(\frac{k_E^2}{(x+y)^2M^2 - xyq^2}\right) + 1\right]^3}$$

$$= \frac{\pi^2}{(2\pi)^4}[(x+y)^2M^2 - xyq^2]^2 \cdot \underbrace{\int_0^{+\infty} dY \frac{Y}{[Y+1]^3}}_{=\ 1/2}$$





> The dominant contribution ~~to~~ the 1-loop integral for $F_2^{(1)}$ is from $k_E \sim \mathcal{O}(M^2)$ or $\mathcal{O}(-q^2)$

So $$F_2^{(1)} = \int_{\Delta} dx\, dy \frac{(Qe)^2}{(4\pi)^2} \frac{\frac{1}{2}(2M)^2\{2(1-x-y)(x+y)\}}{[(x+y)^2M^2 - xyq^2]}$$

$$(x,y) \to (x, z), \quad z = (x+y)$$

$$F_2^{(1)}(q^2=0) = \int_0^1 dz \frac{(Qe)^2}{(4\pi)^2} \frac{(2M)^2}{z^2M^2}(1-z)\, z \times \left(z \leftarrow \int_0^z dx\right)$$

$$= \frac{\alpha_e}{\pi} \times \frac{1}{2}.$$

The anomalous magnetic moment:

$$g = 2 + 2F_2(q^2=0) \simeq 2 + \frac{\alpha_e}{\pi} \quad \text{at 1-loop order.}$$

$$\alpha_e \simeq 1/137$$

It is also possible to compute $\left.\frac{\partial F_2}{\partial q^2}\right|_{q^2=0}$ and higher order

$\approx$ quantum "radius^2" of μ.

§6 Bound States

§6.1 Bethe–Salpeter equation

Consider non-relativistic particles

$$\mathcal{L} = \psi_a^\dagger \left(i\partial_t + \frac{\partial_x^2}{2m_a} - eQ_a\varphi - m_a \right)\psi_a \qquad \left(\begin{array}{l}\text{non-rela. limit} \\ \text{of Dirac fermion} \\ \text{or complex boson}\end{array}\right)$$

Think of $e^- - p^+$, $e^- - \mu^+$, $e^- - e^+$ bound states.

(~~bound states of heavy quark: much the same~~)

(Cooper pair : much the same ; different in details)

Consider

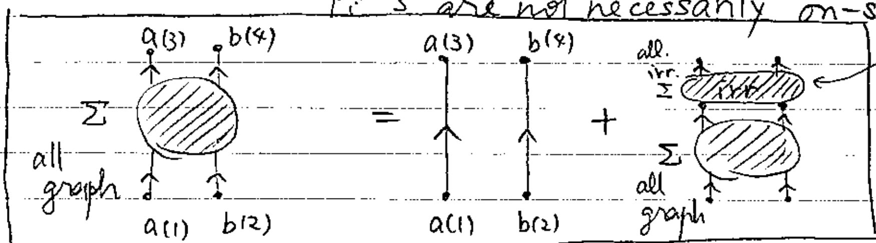
$$\iiiint \langle\Omega| T\{ \psi_a(x_3)\psi_b(x_4)\psi_b^\dagger(x_2)\psi_a^\dagger(x_1)\}|\Omega\rangle\, e^{-ip_1\cdot x_1} e^{-ip_2\cdot x_2} e^{ip_3\cdot x_3} e^{ip_4\cdot x_4}\, d^4x_1\, d^4x_2\, d^4x_3\, d^4x_4$$

$$=: (2\pi)^4\delta^4(p_1+p_2-p_3-p_4)\; G(P_{CM}^\mu ; p^\mu, p'^\mu)$$

Relabel the momenta : $\begin{cases} p_1^\mu = p^\mu + \eta_a P_{CM}^\mu \\ p_2^\mu = -p^\mu + \eta_b P_{CM}^\mu \end{cases}$ $\begin{cases} p_3^\mu = p'^\mu + \eta_a P_{CM}^\mu \\ p_4^\mu = -p'^\mu + \eta_b P_{CM}^\mu . \end{cases}$

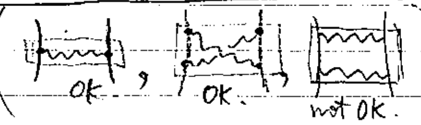
$\left(\eta_a = \frac{m_a}{m_a+m_b} \quad \eta_b = \frac{m_b}{m_a+m_b} . \right)$

p_i^μ's are not necessarily on-shell.



Two-particle irreducible graphs

- external legs are not included.
- remain connected when any one a-particle line and any one b-particle line are cut simultaneously.



$$\boxed{\begin{aligned} G(P_{CM}^\mu ; p^\mu, p'^\mu) &= (2\pi)^4\delta^4(p-p')\, D_a(P_{CM},p')\, D_b(P_{CM},p') \\ &+ D_a(P_{CM},p')\, D_b(P_{CM},p') \int \frac{d^4p''}{(2\pi)^4} K_{irr}(P_{CM}; p', p'')\, G(P_{CM}^\mu ; p^\mu, (p'-p'')^\mu) \end{aligned}}$$

(Bethe–Salpeter eq.)

non-rela parametrization

$$p^0 \Rightarrow \omega \qquad (p')^0 = \omega' \quad ; \quad P_{CM}^0 = (m_a+m_b)+(\Delta E)$$

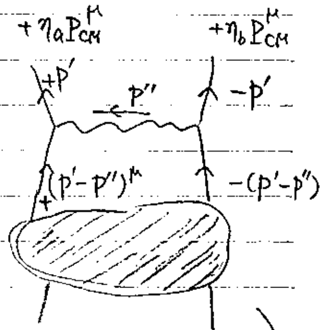
$$p_3^\mu \Rightarrow \left(m_a + \eta_a(\Delta E) + \omega' , \eta_a \vec{P}_{CM} + \vec{p}' \right)$$
$$p_4^\mu \Rightarrow \left(m_b + \eta_b(\Delta E) - \omega' , \eta_b \vec{P}_{CM} - \vec{p}' \right)$$

$$D_a \overset{(\text{tree})}{=} \frac{i}{\left[\eta_a(\Delta E) + \omega' - \frac{(\eta_a \vec{P}_{CM} + \vec{p}')^2}{2m_a} + i\varepsilon \right]}$$

$$D_b \overset{(\text{tree})}{=} \frac{i}{\left[\eta_b(\Delta E) - \omega' - \frac{(\eta_b \vec{P}_{CM} - \vec{p}')^2}{2m_b} + i\varepsilon \right]} .$$

LO approximation to Kirr

in a photon exchange



$$K_{irr} = (-ieQ_a)(-ieQ_b) \frac{(-i)}{\left[(p'')^2 = (\omega'')^2 - (\vec{p}'')^2 \right]} .$$

in a phonon exchange

$$\left(K_{irr} = \left(\frac{ig}{\Lambda}\right)^2 \frac{(+i) \quad (\vec{p}'' \cdot \vec{p}'')}{\left[(\omega'')^2 - v_s^2 (\vec{p}'')^2 + i\varepsilon \right]} . \longleftarrow \mathcal{L}_{int} = \frac{g}{\Lambda}(\partial \cdot \vec{\varphi}) \psi^\dagger \psi \right)$$

Now, think of a case there are contributions of the form

$$G(P_{CM}^\mu ; p^\mu , p'^\mu) = \sum_n \left\{ \chi_n(p') \frac{i}{(P_{CM})^2 - M_n^2 + i\varepsilon} \chi_n^*(p) \right\} + \left(\begin{matrix}\text{non-}\\ \text{pole}\\ \text{terms}\end{matrix} \right) .$$

$\Longleftrightarrow$ $\exists$ bound states

$$\boxed{\begin{aligned} &\langle \Omega | T\{ \psi_a(p_3) \psi_b(p_4) \} | n ; \vec{P}_{CM*} \rangle \\ &= (2\pi)^4 \delta^3(\vec{P}_{CM} - \vec{P}_{CM*}) \, \delta\left(m_a + m_b + (\Delta E) - E_{n, \vec{P}_{CM*}} \right) \cdot \chi_n(p') \end{aligned}}$$

Q: Verify that

$$\begin{cases} \left[\langle \Omega | \psi(x) \psi(x) \psi^\dagger(x) \psi^\dagger(x) | \Omega \rangle \right] = +6 \\ \left[G(P_{CM}^\mu ; p^\mu , p'^\mu) \right] = -6 \\ \left[\chi_n(p') \right] = -2 . \end{cases}$$

$$\begin{cases} \left[\langle \Omega | \psi(x) \psi(x) | \text{state} \rangle \right] = +2 \\ \left[\langle \Omega | \psi(p) \psi(p) | \text{state} \rangle \right] = -6 \end{cases}$$

No. both the RHS & LHS of the eqn above have mass-dim -6 (sanity check)

Comparing the residue on a pole in the BS equation, we obtain

$$\chi_n(p') \cong D_a(\underline{\underline{\Delta E}}, \vec{P}_{CM}; p')\, D_b(\underline{\underline{\Delta E}}, \vec{P}_{CM}; p') \times \int \frac{d^3p''}{(2\pi)^3} \int \frac{d\omega''}{2\pi} \frac{i(eQ_a eQ_b)}{(\omega'')^2 - |\vec{p}''|^2} \chi_n(p'-p'') \qquad (*)$$

$$\boxed{\text{Suppose that we can ignore } (\omega'')^2 \text{ against } |\vec{p}''|^2 \text{ in the dominant region of the integral (verified later).}} \qquad (**)$$

Then we see that the $[\omega' = (p')^0]$-dependence in the RHS of (*) comes only from $D_a \cdot D_b$. So,

$$\chi_n(\omega', \vec{p}'; \vec{P}_{CM}) = \frac{i}{\left[\eta_a (\underline{\underline{\Delta E}})_n + \omega' - \frac{(\eta_a \vec{P}_{CM} + \vec{p}')^2}{2m_a} + i\varepsilon\right]} \frac{i \times \underline{\underline{\chi}}_n(\vec{p}'; \vec{P}_{CM})}{\left[\eta_b (\underline{\underline{\Delta E}}_n) - \omega' - \frac{(\eta_b \vec{P}_{CM} - \vec{p}')^2}{2m_b} + i\varepsilon\right]},$$

$$\psi_n(\vec{p}'; \vec{P}_{CM}) := \int \frac{d\omega'}{2\pi} \chi_n(\omega', \vec{p}'; \vec{P}_{CM})$$

$$= \frac{-\underline{\chi}_n(\vec{p}'; \vec{P}_{CM})}{2\pi} \frac{1}{\left[(\underline{\underline{\Delta E}})_n - \frac{\vec{P}_{CM}^2}{2(m_a+m_b)} - \frac{|\vec{p}'|^2}{2\mu_{ab}}\right]} \int d\omega' \left(\frac{1}{\left[\eta_a \underline{\underline{\Delta E}}_n + \omega' - \frac{(\ )^2}{2m_a} + i\varepsilon\right]} + \frac{1}{\left[\eta_b (\underline{\underline{\Delta E}}_n) - \omega' - \frac{(\ )^2}{2m_b} + i\varepsilon\right]} \right)$$

$$= \frac{-(-2\pi i)}{2\pi} \frac{\underline{\chi}_n(\vec{p}'; \vec{P}_{CM})}{[\quad]} = \frac{(+i)}{\left[(\underline{\underline{\Delta E}})_n - \frac{\vec{P}_{CM}^2}{2(m_a+m_b)} - \frac{\vec{p}'^2}{2\mu_{ab}}\right]} \underline{\chi}_n(\vec{p}', \vec{P}_{CM}).$$

Thus, the eqn (*) can be rewritten as

$$\int \frac{d^3p''}{(2\pi)^3} \frac{(eQ_a eQ_b)}{|\vec{p}''|^2} \psi_n(\vec{p}' - \vec{p}''; \vec{P}_{CM}) \overset{(*)}{\cong} \frac{i}{D_a D_b} \chi_n(p') = i\, \underline{\chi}_n = \left(\underline{\underline{\Delta E}}_n - \frac{\vec{P}_{CM}^2}{2(m_a+m_b)} - \frac{(\vec{p}')^2}{2\mu_{ab}} \right) \psi_n(\vec{p}'; \vec{P}_{CM})$$

$$\Big\Downarrow \text{ Fourier transform in } \vec{p}' \to \vec{r}$$

$$\boxed{\left((\underline{\underline{\Delta E}})_n - \frac{\vec{P}_{CM}^2}{2(m_a+m_b)} \right) \tilde{\psi}_n(\vec{r}; \vec{P}_{CM}) = \left(\frac{-\vec{\partial}_r^2}{2\mu_{ab}} + \frac{e^2 Q_a Q_b}{4\pi r} \right) \tilde{\psi}_n(\vec{r}; \vec{P}_{CM})}$$

Schrödinger equation

$$\left(\frac{1}{\mu_{ab}} := \frac{1}{m_a} + \frac{1}{m_b} = \left(\frac{m_a m_b}{(m_a+m_b)} \right)^{-1} \text{ reduced mass} \right)$$

$$\boxed{\chi_n(p') \text{ "is" the matrix element } \langle 0 | T\{\psi_a \psi_b\} | \text{bound state}_n \rangle}$$

(notation: χ_n, ψ_n as in LL4 or Takahashi; $\underline{\chi}_n \sim \Gamma$ two subscripts in LL4)

§ 6.2 Hydrogen atom spectroscopy in QED

Schrödinger eq: $\Delta E_n = -\frac{m_e \alpha^2}{2n^2}$

But....
- QED corrections to $\quad \psi^\dagger_{(e)}\left(i\partial_t - m + \frac{\vec{\partial}^2}{2m} - eQ_e\varphi\right)\psi_{(e)}$ $\longrightarrow$ fine structure.
- proton also moves $\longrightarrow$ hyperfine structure.
- Kirr is not just (photon line) $\longrightarrow$ Lamb shift

§ 6.2.1 Fine structure

electron Lagrangian

$$\mathcal{L} = \bar{\Psi}\left\{i\gamma^\mu(\partial_\mu + ieQ_eA_\mu) - m_{(e)}\right\}\Psi, \qquad \gamma^0 = \begin{pmatrix} \mathbb{1} & \\ & -\mathbb{1}\end{pmatrix}, \quad \gamma^i = \begin{pmatrix} & \tau^i \\ -\tau^i & \end{pmatrix} \Rightarrow i\not{\partial} \sim \begin{pmatrix} E & -\vec{p}\cdot\vec{\tau} \\ \vec{p}\cdot\vec{\tau} & -E\end{pmatrix}$$

$\Rightarrow$ diagonalize by $\vec{p}$-dependent field redefinition.

$$\Rightarrow \mathcal{L} \cong (\psi^\dagger_{e^-}, \psi^\dagger_{e^+})\begin{bmatrix} i\partial_t - m - eQ_e\varphi + \frac{(\vec{\partial})^2}{2m_e} + \frac{eQ_e(\vec{\partial}\cdot\vec{E})}{8m_e^2} + \frac{eQ_e(\vec{E}\times(-i\vec{\partial}))\cdot\vec{\tau}}{4m_e^2} + \frac{(\vec{\partial})^4}{8m_e^3} + \cdots & ☆ \\ ☆ & i\partial_t + m - eQ_e\varphi - \frac{(\vec{\partial})^2}{2m_e} + \frac{eQ_e(\vec{\partial}\cdot\vec{E})}{8m_e^2} - \frac{eQ_e(\vec{E}\times(-i\vec{\partial}))\cdot\vec{\tau}}{4m_e^2} + \cdots \end{bmatrix}$$

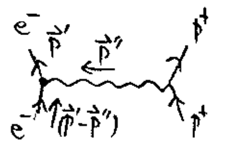
$$\times \begin{pmatrix}\psi_{e^-} \\ \psi_{e^+}\end{pmatrix} \qquad (\psi_{e^-}, \psi_{e^+} : \text{both 2-component spinor fields.})$$

[see homework V-1 for more details] $\qquad \left(☆\text{'s} \sim \mathcal{O}\left(\frac{1}{m_e^2}\right)\right)$

Corrections:

$$D_{e^-} = \frac{i}{\left[\eta_e(\Delta E) + \omega' - \frac{(\eta_a\vec{p}_{CM} + \vec{p}')^2}{2m_e} - \frac{(\eta_a\vec{p}_{CM} + \vec{p}')^4}{8m_e^3} + i\varepsilon\right]}$$

$$K_{irr.} = (-ieQ_{p^+})(-ieQ_{e^-})\left\{1 + \frac{-|\vec{p}''|^2}{8m_e^2} + \frac{i(\vec{p}''\times(\vec{p}'-\vec{p}''))\cdot\left(\frac{\vec{\tau}}{2}\right)}{2m_e^2} + \cdots\right\}$$



At $\vec{p}_{CM} = \vec{0}$

$$(\Delta E_n)\tilde{\psi}_n(\vec{r}; \vec{p}_{CM} = \vec{0}) \cong \left\{\frac{-\vec{\partial}_r^2}{2\mu_{ab}} - \frac{\vec{\partial}_r^4}{8m_e^3} + \left(\frac{e^2(Q_{e^-}Q_{p^+})}{4\pi r}\right) - \frac{(eQ_{e^-})(\vec{\partial}\cdot\vec{E})}{8m_e^2} - \frac{(\vec{E}\times(-i\vec{\partial}))\cdot\frac{\vec{\tau}}{2}}{2m_e^2}\right\}\tilde{\psi}_n(\vec{r}; \vec{p}_{CM}$$

At the leading order (Schrödinger eq.) $r \sim 1/m_e\alpha \quad p \sim m_e\alpha$.

$\Rightarrow$ correction terms to (ΔE_n) are of order $(m_e \cdot \alpha^4)$.

$\vec{L}\cdot\vec{S}$ coupling $\longrightarrow$ (L^2, S^2, J^2, J_z) eigenstates

§ 6.2.2 Hyperfine structure

Let us now include $(\varphi, -\vec{A}) = A_\mu$ exchange not just φ.

$$\mathcal{L} = \psi_a^\dagger \left\{ i\partial_t - m_a - eQ_a\varphi + \frac{(\vec{\partial}_i - ieQ_a\vec{A}_i)^2}{2m_a} - \left(\frac{g_a}{2}\right)\left(\frac{\tau^i}{2}\right)\frac{(\epsilon^{ijk}\partial_j A_k)e}{m_a} + \cdots \right\} \psi_a$$

$(g_a/2) = -Q_e$ for e^-. $\quad (g_a/2)$ for p^+ : just keep it as a parameter.
$\quad\;\; =+1$

In addition to

$$K_{irr}^{LO} \simeq (-ieQ_{e^-})(-ieQ_{p^+})\frac{(-i)(+1)}{(-|\vec{q}|^2)}, \quad \text{now we have}$$

$$\Delta K_{irr} \cong \frac{(-i)\times(-1)}{-|\vec{q}|^2} \times \left((ieQ_{e^-})\frac{(\vec{p}_{e,in} + \vec{p}_{e,out})}{m_e} + \frac{eg_e}{2m_e}(\vec{S}_e \times \vec{q})\right) \cdot \left((ieQ_{p^+})\frac{(\vec{p}_{p,in} + \vec{p}_{p,out})}{m_p} + \frac{eg_p}{2m_p}(\vec{S}_p \times (-\vec{q}))\right).$$

- Two terms out of the four terms : don't do much (just correction) (spherical or $\vec{S}_e \cdot \vec{L}$).
- Two other terms : in terms of "potential"

$$\frac{\alpha}{r} \cdot \frac{(\vec{q} \times \vec{p}_e)}{m_e m_p} \cdot \vec{S}_p \sim \frac{\alpha}{r^3}\frac{\vec{L} \cdot \vec{S}_p}{m_e m_p} \quad \text{and} \quad e^2 \delta^3(\vec{r}) \frac{\vec{S}_e \cdot \vec{S}_p}{m_e m_p}$$

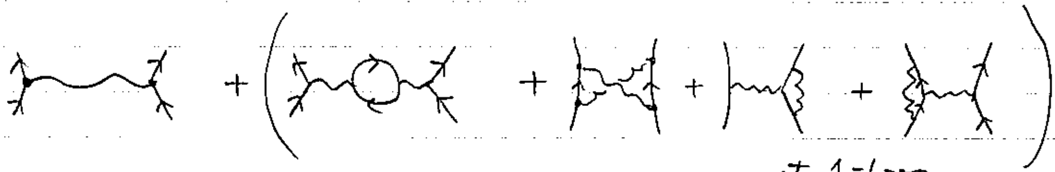
$$\text{extra}(\Delta E) \sim \frac{\alpha}{m_e m_p} \times (m_e\alpha)^3 \sim \frac{m_e^2}{m_p}\alpha^4.$$

Hydrogen atom in the 1s state : 21cm line.
$\left(\begin{array}{l} = 1.4 \text{ GHz} \\ = 5.9 \times 10^{-6} \text{eV} \end{array} \right)$.

§ 6.2.3 Lamb shift

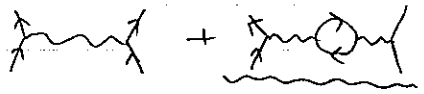
So far just one 2PI graph  has been taken into account.

But...

K_{irr} = 

at 1-loop.

+ higher loop contributions...

☆  modifies

$$K_{irr} \simeq \frac{(-ieQ_{e^-})(-ieQ_{p^+})\, i}{|\vec{q}|^2} \rightsquigarrow (-ieQ_{e^-})(-ieQ_{p^+})\left(\frac{i}{|\vec{q}|^2} + i\frac{\alpha}{15\pi}\frac{1}{m_e^2}\right)$$

(approximation at $|\vec{q}| \ll m_e$. (say $|\vec{q}| \sim \mathcal{O}(m_e\alpha)$))

need 1-loop computation, which has not been covered in this course yet.

$\Longrightarrow$ modifies the Schrödinger eq. by

$$\frac{\alpha}{r} \longrightarrow \left(\frac{\alpha}{r} + \frac{4}{15}\frac{\alpha^2}{m_e^2}\delta^3(\vec{r})\right)$$

(ΔE) changes by $\mathcal{O}(m_e\alpha^5/\pi)$

☆ three other graphs : subtle treatment required.

[see Landau-Lifshitz vol.4 (QED) §123.]

still. (ΔE) change by $\mathcal{O}\left(m_e\frac{\alpha^5}{\pi}\times\ln(1/\alpha)\right)$

Summary

$\boxed{(\Delta E) \sim (m_e\alpha^2)}$

non-rela. Schrödinger.

→ (e^- relativistic correction) (φ-exchange)

(fine structure)

$\boxed{\text{extr } m_e\alpha^2 \times (\beta_{(e)}^2 \sim \alpha^2) \sim m_e\alpha^4}$

→ 1-loop correction to K_{irr}.

extra

$\boxed{m_e\alpha^4 \times \frac{\alpha}{\pi}\ln(1/\alpha)}$

(Lamb shift)

→ ($\vec{A}$-exchange) [relativistic corr. both e^- & p^+]

(hyperfine structure)

$\boxed{\text{extra } m_e\alpha^2 \times (\beta_{(e)}\cdot\beta_{(p)}) \sim m_e\alpha^2\cdot\left(\alpha\cdot\frac{m_e}{m_p}\alpha\right)}$

$\sim \boxed{\frac{m_e^2}{m_p}\alpha^4}$